The Turing Machine

Algorithms and Complexity Theory

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Preamble

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- How do we express that a problem is **harder** than another problem ?
- Is it possible to construct a **hierarchy** of problems, based on hardness ?

How do we formally express that a problem is solvable ?

Part I: What is a problem ?

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Problem instance: *Example:* $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 2), (1, 3), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

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In what follows, we will deal with **decision problems only** !

Decision vs optimisation problems

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Claim

An optimisation problem[a] is as hard to solve as it's equivalent decision problem[b].

[a] such as MAX-CLIQUE
[b] in our example, k-CLIQUE

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  - Repeat until we get a **yes** or $k = 0$. The answer to MAX-CLIQUE is k.

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If k-CLIQUE can be solved in $O(n^p)$ time then MAX-CLIQUE can be solved in $O(n^{p+1})$ time

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In Complexity Theory, we will further agree to **ignore** the degrees of polynomials.

Problems vs algorithms

Definition (Algorithm)

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- *How do we find a simple mathematical model that captures all these ways to compute? [1]*

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- *How do we find a simple mathematical model that captures all these ways to compute? [1]*
- *How do we formally specify **algorithms** ?*

Part II: *The Turing Machine*

A **deterministic Turing Machine** (DTM) is a tuple $M = (K, F, \Sigma, \delta, s)$ where:

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We assume *data* is *read/written* on a **tape**.

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Σ={0,1,_,>}

The tape contains **cells**, and each cell contains **symbols**. Σ is the set of symbols. Σ always contains _ (the blank symbol) and $>$ (the first symbol).

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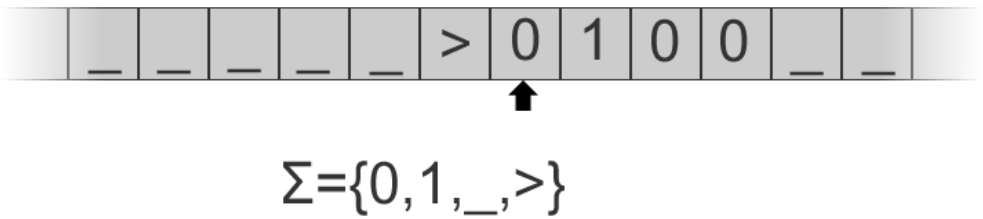

A **tape head** indicates the current position on the tape.

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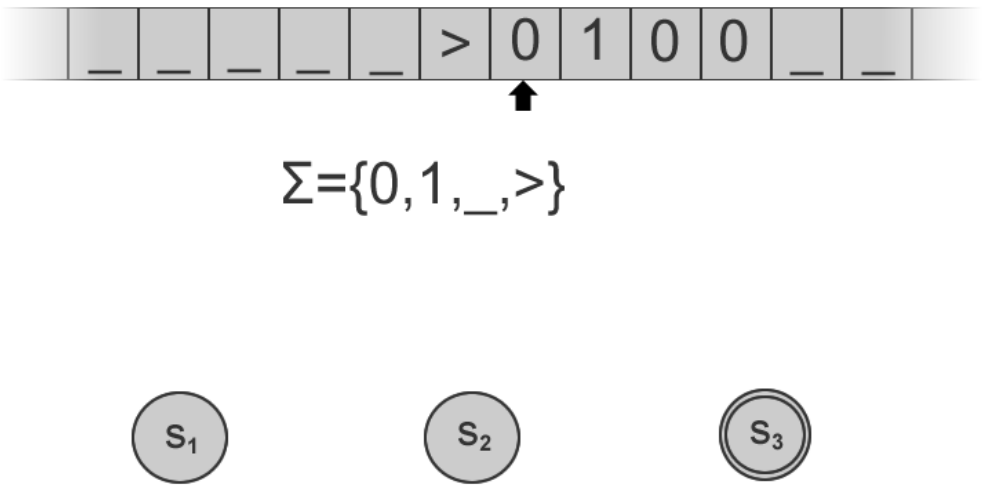

A set of **states** K and a set of **accepting states** $F \subseteq K$. Here $K = \{s_1, s_2, s_3\}$ and $F = \{s_3\}$.

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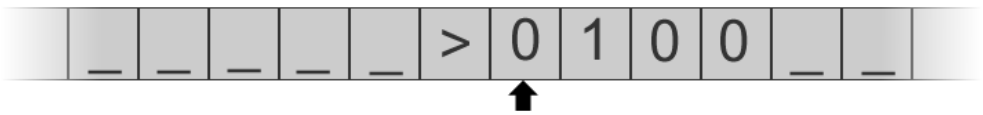

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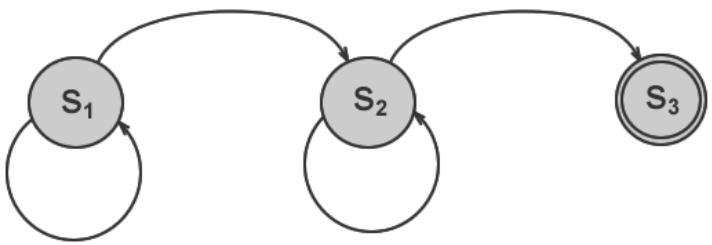

A **transition function** $\delta : K \times \Sigma \rightarrow K \times \Sigma \times \{L, H, R\}$

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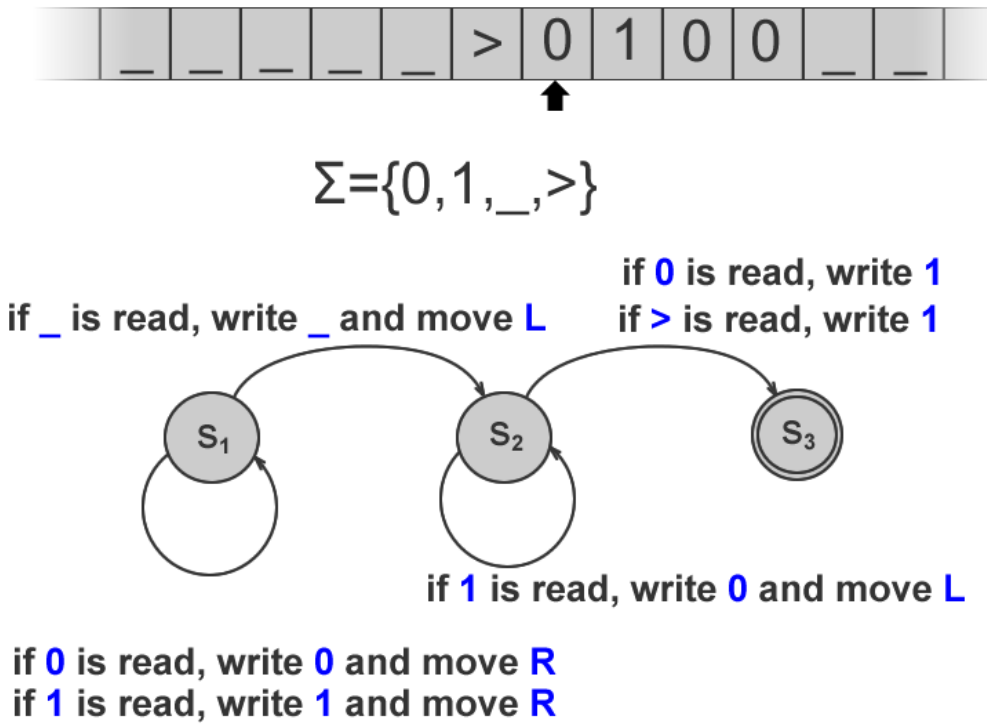

$\delta : K \times \Sigma \to K \times \Sigma \times \{L, H, R\}$

Write down δ for our example

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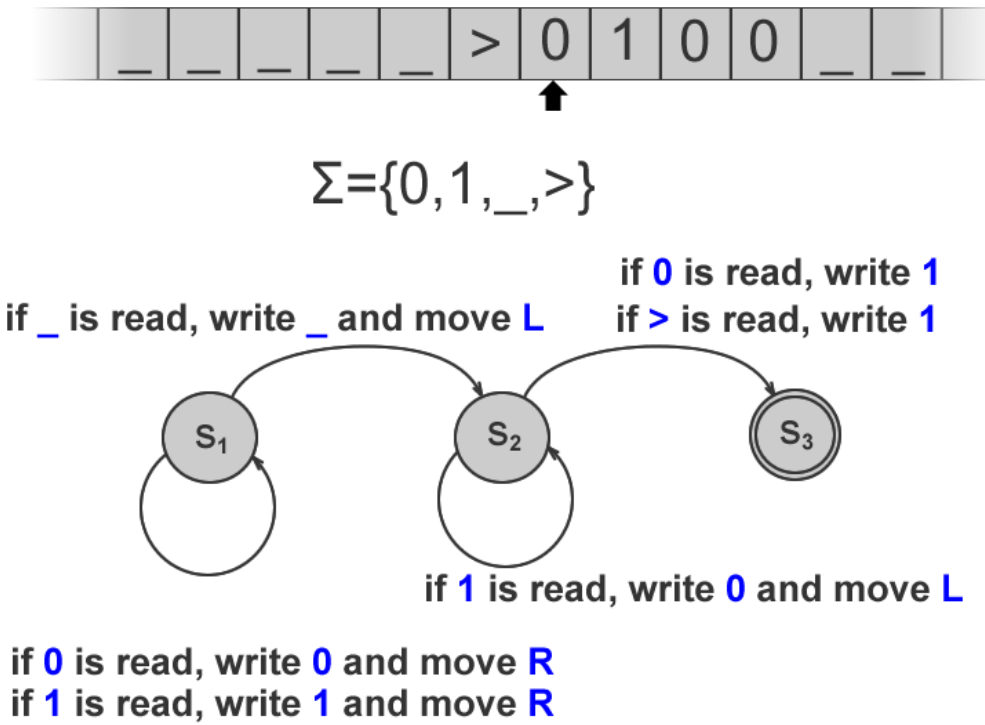

An **initial state** s. In our case $s = s_1$.

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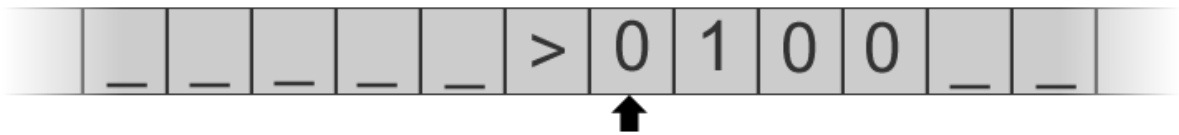
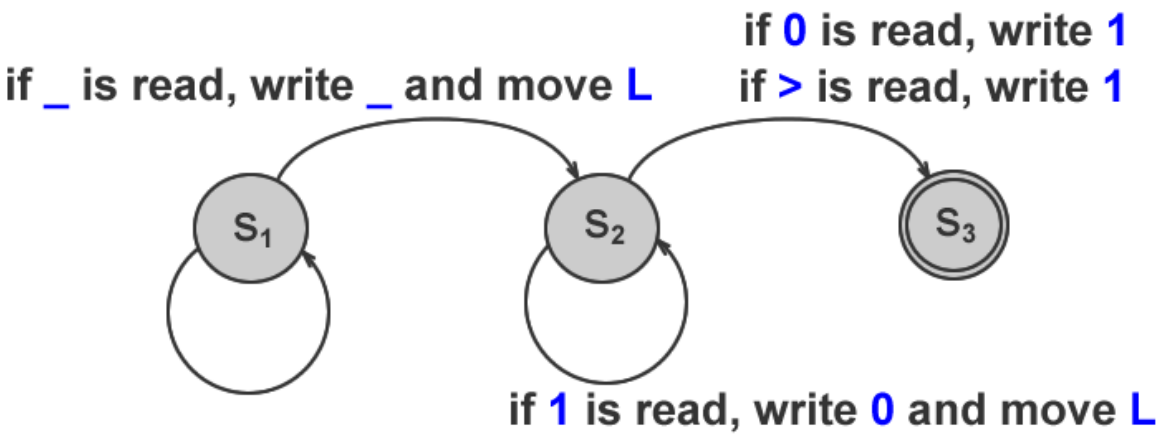
- We also write $M(w)$ to refer to the output of the Turing Machine M, on string w.

Example

What does our machine do ?



$\Sigma = \{0, 1, \_, >\}$



if 0 is read, write 0 and move R
if 1 is read, write 1 and move R

Another example: `http://www.youtube.com/watch?v=WJ-ODmFjmrU&feature=relmfu`

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Definition (Running time [1])

Let $P : I \to \{0, 1\}$, $T : \mathbb{N} \to \mathbb{N}$, and M be a Turing Machine. We say M solves P if for every encoding $w \in \Sigma^*$ of a problem instance from I, if the tape is initialized with w, then it **halts** with $P(w)$ written as output. If the computation takes **at most** $T(|w|)$ steps **for every** w, then we say the running time of M is $T(n)$.

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Definition

If M is a Turing Machine with running time $T(n)$, which solves the problem P, we say that P is **solvable** in time $T(n)$

Questions (you should ask):

[1] Recall our assumption regarding *distinction* between running times !

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- *Is there no other formalism capable of doing computations faster*[1] *?*

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Encoding does not matter

Proposition

Every problem $P : I \to \{0, 1\}$ *that is* ***solvable*** *in time* $T(n)$ *by a Turing Machine with alphabet* Σ *(which is used for encoding problem instances in* I*), is also solvable in time* $O(\log(|\Sigma|)) \cdot T(n)$ *using the alphabet* $\{0, 1, >, \_\}$.

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Blackboard !

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- software $\leftrightarrow$ hardware $\leftrightarrow$ data

Universal Turing Machines

Definition (Efficient Turing Machine [1])

For every Turing Machine M having running time $T(n)$, and every word $w \in \{0,1\}$ there exists a **Universal Turing Machine** U, such that $U(enc(M), w) = M(w)$.
Moreover, the **running time of** U is $c \cdot T(n)$, where c only depends on M, and not on the size of w.

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- We encode s as $|K| + |F| + 1$ and L, H, R as $|K| + |F| + i$, with $i \in \{2, 3, 4\}$

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- We encode s as $|K| + |F| + 1$ and L, H, R as $|K| + |F| + i$, with $i \in \{2, 3, 4\}$

Each integer x will be represented as a string $enc(x)$ using $\lceil \log(|K| + |F| + 4) \rceil$ **bits**

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enc(δ) is a **sequence** of transitions, encoded as above

Turing Machines and computability

Claim (Church)

For every program $P : \Sigma^* \to \{0, 1\}$, written in any programming language, and with running time $T(n)$, there exists an ***equivalent*** *Turing Machine with running time $O(T(n)^p)$, where p is a* ***fixed*** *natural number.*

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- For this reason, we convene to describe Turing Machines in **pseudocode**, for simplicity and clarity.

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*For every program $P : \Sigma^* \to \{0, 1\}$, written in any programming language, and with running time $T(n)$, there exists an **equivalent** Turing Machine with running time $O(T(n)^p)$, where p is a **fixed** natural number.*

- Evidence: **Lambda Calculus**, **RAM**, **μ-recursive functions**
- For this reason, we convene to describe Turing Machines in **pseudocode**, for simplicity and clarity.

Example - blackboard

Part III: *Is finding the correct answer **harder** than checking if **a given answer** is correct ?*

Nondeterministic Turing Machines

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We need a tool to formally express **exhaustive search**

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Nondeterministic Turing Machine A non-deterministic Turing Machine (NTM) is a TM $M = (K, F, \Sigma, \delta, s)$, where δ is a **relation**: $\delta \subset K \times \Sigma \times K \times \Sigma \times \{L, H, R\}$

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NTM's as **pseudocode**: (`choice`, `success`, `fail`) - blackboard

NTMs vs DTMs

Proposition

*For every NTM M', there exists an **equivalent** DTM M. If M' runs in polynomial time, M runs in exponential time.*

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P = NP (?!)

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