Chapter 5

State Feedback

5.1 Introduction

The state of a dynamical system is a collection of variables that permits prediction of the future development of a system. It is therefore very natural to base control on the state. This will be explored in this chapter. It will be assumed that the system to be controlled is described by a state model. Furthermore it is assumed that the system has one control variable. The technique which will be developed may be viewed as a prototype of an analytical design method. The feedback control will be developed step by step using one single idea, the positioning of closed loop eigenvalues in desired locations.

The case when all the state variables are measured is first discussed in Section 5.3. It is shown that if the system is reachable then it is always possible to find a feedback so that the closed loop system has prescribed eigenvalues.

In Section 5.4 we consider the problem of determining the states from observations of inputs and outputs. Conditions for doing this are established and practical ways to do this are also developed. In particular it is shown that the state can be generated from a dynamical system driven by the inputs and outputs of the process. Such a system is called an state estimator or observer. The observer can be constructed in such a way that its state approaches the true states with dynamics having prescribed eigenvalues. It will also be shown that the problem of finding an observer with prescribed dynamics is mathematically equivalent to the problem of finding a state feedback.

In Section 5.5 it is shown that the results of Sections 5.3 and 5.4 can be combined to give a controller based on measurements of the process output only. The conditions required are simply that the system is reachable and observable. This result is important because the controller has a very interesting structure. The controller contains a mathematical model of the system to be controlled. This is called the internal model principle. The solution to the eigenvalue assignment problem also illustrates that the notions of reachability and observability are essential. The result gives a good interpretation of dynamic feedback. It shows that the dynamics of the controller arises from the need to reconstruct the state of the system.

Finally in Section **??** we give an example that illustrates the design technique.

The details of the analysis and designs in this chapter are carried out for systems with one input and one output. It turns out that the structure of the controller and the forms of the equations are exactly the same for systems with many inputs and many outputs. There are also many other design techniques that give controllers with the same structure. A characteristic feature of a controller with state feedback and an observer is that the complexity of the controller is given by the complexity of the system to be controlled. The controller actually contains a model of the system, the internal model principle.

5.2 Reachability

We begin by disregard the output masurements and focus on the evolution of the state which is given by

$$\frac{dx}{dt} = Ax + Bu, \tag{5.1}$$

where the system is assumed to be of order n. A fundamental question is if it is possible to find control signals so that any point in the state space can be reached. For simplicity we assume that the initial state of the system is zero.

We will first provide an heuristic argument based on formal calculations with delta functions. When the initial state is zero The response of the state to a unit step in the input is given by

$$x(t) = \int_0^t e^{A(t-\tau)} B d\tau = A^{-1}(e^{At} - I)B \tag{5.2}$$

The derivative of a unit step function is the delta function $\delta(t)$ which may be regarded as a function which is zero everywhere except at the origin with the property that

$$\int_{\infty}^{\infty} \delta(t)dt = 1$$

The response of the system to a delta function is thus given by the derivative of (5.2)

$$\frac{dx}{dt} = e^{At}B$$

The response to the derivative of a delta function is thus

$$\frac{dx}{dt} = Ae^{At}B$$

The input

$$u(t) = \alpha_1\delta(t) + \alpha_2\dot{\delta}(t) + \alpha_d\dot{\delta}^2(t) + \cdots + \alpha_n\delta^{n-1}(t)$$

thus gives the state

$$x(t) = \alpha_1 e^{At}B + \alpha_2 Ae^{At}B + \alpha_3 A^2 e^{At}B + \cdots + \alpha_n A^{n-1}e^{At}B$$

Hence

$$x(0+) = \alpha_1 B + \alpha_2 AB + \alpha_3 A^2 B + \cdots + \alpha_n A^{n-1}B$$

The right hand is a linear combination of the columns of the matrix.

$$W_r = B \quad AB \quad \ldots \quad A^{n-1}B \tag{5.3}$$

To reach an arbitrary point in the state space it must thus be required that there are n linear independent columns of the matrix W_c. The matrix is therefor called the reachability matrix.

An input consisting of a sum of delta functions and their inputs is a very violent signal. To see that an arbitrary point can be reached with smoother signals we can also argue as follow. Assuming that the initial condition is zero the state of the system is given by

$$x(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \int_0^t e^{A(\tau)}Bu(t-\tau)d\tau$$

It follows from the theory of matrix functions that

$$e^{A\tau} = I\alpha_0(s) + A\alpha_1(s) + \ldots + A^{n-1}\alpha_{n-1}(s)$$

and we find that

$$x(t) = B\int_0^t \alpha_0(\tau)u(t-\tau)d\tau + AB\int_0^t \alpha_1(\tau)u(t-\tau)d\tau +$$
$$\ldots + A^{n-1}B\int_0^t \alpha_{n-1}(\tau)u(t-\tau)d\tau$$

Again we observe that the right hand side is a linear combination of the columns of the reachability matrix W_r given by (5.3).

We illustrate by two examples.

Example 9 *(Reachability of the Inverted Pendulum).* The linearized model of the inverted pendulum is derived in Example **??**. The dynamics matrix and the control matrix are

$$A = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}, \quad B = \begin{matrix} 0 \\ 1 \end{matrix}$$

The reachability matrix is

$$W_r = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \tag{5.4}$$

This matrix has full rank and we can conclude that the system is reachable.

Example 10 *(System in Reachable Canonical Form).* Next we will consider a system by in *reachable canonical form*:

$$\frac{dz}{dt} = \begin{matrix} -a_1 & -a_2 & \ldots & a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{matrix} z + \begin{matrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix} u = \tilde{A}z + \tilde{B}u$$

The inverse of the reachability matrix is

$$\tilde{W}_r^{-1} = \begin{matrix} 1 & a_1 & a_2 & \ldots & a_n \\ 0 & 1 & a_1 & \ldots & a_{n-1} \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 1 \end{matrix} \tag{5.5}$$

To show this we consider the product

$$\tilde{B} \quad \tilde{A}\tilde{B} \quad \cdots \quad \tilde{A}^{n-1}BW_r^{-1} = w_0 \quad w_1 \quad \cdots \quad w_{n-1}$$

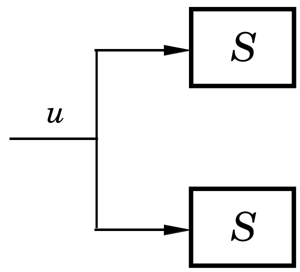

Figure 5.1: A non-reachable system.

where

$$
\begin{aligned}
w_0 &= \tilde{B} \\
w_1 &= a_1\tilde{B} + \tilde{A}\tilde{B} \\
&\vdots \\
w_{n-1} &= a_{n-1}B + a_{n-2}\tilde{A}B + \cdots + \tilde{A}^{n-1}B
\end{aligned}
$$

The vectors w_k satisfy the relation

$$w_k = a_k + \tilde{w}_{k-1}$$

Iterating this relation we find that

$$
\begin{matrix} w_0 & w_1 & \cdots & w_{n-1} \end{matrix} = \begin{matrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \end{matrix}
$$

which shows that the matrix (5.5) is indeed the inverse of $\tilde{W}_r$.

Systems That are Not Reachable

It is useful of have an intuitive understanding of the mechanisms that make a system unreachable. An example of such a system is given in Figure 5.1. The system consists of two identical systems with the same input. The intuition can also be demonstrated analytically. We demonstrate this by a simple example.

Example 11 *(Non-reachable System).* Assume that the systems in Figure 5.1 are of first order. The complete system is then described by

$$\frac{dx_1}{dt} = -x_1 + u$$
$$\frac{dx_2}{dt} = -x_2 + u$$

The reachability matrix is

$$W_r = \begin{matrix} 1 & -1 \\ 1 & -1 \end{matrix}$$

This matrix is singular and the system is not reachable.

Coordinate Changes

It is interesting to investigate how the reachability matrix transforms when the coordinates are changed. Consider the system in (5.1). Assume that the coordinates are changed to $z = Tx$. It follows from linear algebra that the dynamics matrix and the control matrix for the transformed system are

$$\tilde{A} = TAT^{-1}$$
$$\tilde{B} = TB$$

The reachability matrix for the transformed system then becomes

$$\tilde{W}_r = \tilde{B} \quad \tilde{A}\tilde{B} \quad \ldots \quad \tilde{A}^{n-1}\tilde{B} =$$

We have

$$\begin{aligned} \tilde{A}\tilde{B} &= TAT^{-1}TB = TAB \\ \tilde{A}^2\tilde{B} &= (TAT^{-1})^2 TB = TAT^{-1}TAT^{-1}TB = TA^2B \\ &\vdots \\ \tilde{A}^n\tilde{B} &= TA^nB \end{aligned}$$

and we find that the reachability matrix for the transformed system has the property

$$\tilde{W}_r = \tilde{B} \quad \tilde{A}\tilde{B} \quad \ldots \quad \tilde{A}^{n-1}\tilde{B} = TB \quad AB \quad \ldots \quad A^{n-1}B = TW_r \tag{5.6}$$

This formula is very useful for finding the transformation matrix T.

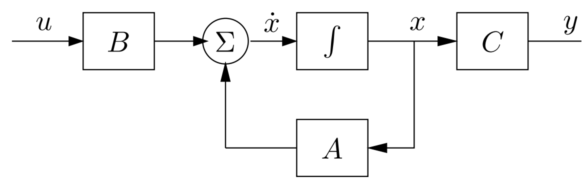

Figure 5.2: Block diagram of the process described by the state model in Equation (5.7).

5.3 State Feedback

Consider a system described by the linear differential equation

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu \\ y &= Cx\end{aligned} \tag{5.7}$$

A block diagram of the system is shown in Figure 5.2. The output is the variable that we are interested in controlling. To begin with it is assumed that all components of the state vector are measured. Since the state at time t contains all information necessary to predict the future behavior of the system, the most general time invariant control law is function of the state, i.e.

$$u(t) = f(x)$$

If the feedback is restricted to be a linear, it can be written as

$$u = -Kx + K_r r \tag{5.8}$$

where r is the reference value. The negative sign is simply a convention to indicate that negative feedback is the normal situation. The closed loop system obtained when the feedback (5.7) is applied to the system (5.8) is given by

$$\frac{dx}{dt} = (A - BK)x + BK_r r \tag{5.9}$$

It will be attempted to determine the feedback gain K so that the closed loop system has the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \ldots + p_{n-1}s + p_n \tag{5.10}$$

This control problem is called the eigenvalue assignment problem or the pole placement problem (we will define “poles” more formally in a later chapter).

Examples

We will start by considering a few examples that give insight into the nature of the problem.

Example 12 *(The Double Integrator).* The double integrator is described by

$$
\begin{aligned}
\frac{dx}{dt} &= \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} x + \begin{matrix} 0 \\ 1 \end{matrix} u \\
y &= \begin{matrix} 1 & 0 \end{matrix} x
\end{aligned}
$$

Introducing the feedback

$$u = -k_1 x_1 - k_2 x_2 + K_r r$$

the closed loop system becomes

$$
\begin{aligned}
\frac{dx}{dt} &= \begin{matrix} 0 & 1 \\ -k_1 & -k_2 \end{matrix} x + \begin{matrix} 0 \\ K_r \end{matrix} r \\
y &= \begin{matrix} 1 & 0 \end{matrix} x
\end{aligned}
\tag{5.11}
$$

The closed loop system has the characteristic polynomial

$$det \begin{matrix} s & -1 \\ k_1 & s + k_2 \end{matrix} = s^2 + k_2 s + k_1$$

Assume it is desired to have a feedback that gives a closed loop system with the characteristic polynomial

$$p(s) = s^2 + 2\zeta\omega_0 s + \omega_0^2$$

Comparing this with the characteristic polynomial of the closed loop system we find find that the feedback gains should be chosen as

$$k_1 = \omega_0^2, \quad k_2 = 2\zeta\omega_0$$

To have unit steady state gain the parameter K_r must be equal to $k_1 = \omega_0^2$. The control law can thus be written as

$$u = k_1(r - x_1) - k_2 x_2 = \omega_0^2(r - x_1) - 2\zeta_0\omega_0 x_2$$

In the next example we will encounter some difficulties.

Example 13 *(An Unreachable System).* Consider the system

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = Cx = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

with the control law

$$u = -k_1 x_1 - k_2 x_2 + K_r r$$

The closed loop system is

$$\frac{dx}{dt} = \begin{bmatrix} -k_1 & 1 - k_2 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} K_r \\ 0 \end{bmatrix} r$$

This system has the characteristic polynomial

$$\det \begin{bmatrix} s + k_1 & -1 + k_2 \\ 0 & s \end{bmatrix} = s^2 + k_1 s = s(s + k_1)$$

This polynomial has zeros at $s = 0$ and $s = -k_1$. One closed loop pole is thus always equal to $s = 0$ and it is not possible to obtain an arbitrary characteristic polynomial.

This example shows that the pole placement problem cannot be solved. An analysis of the equation describing the system shows that the state x_2 is not reachable. It is thus clear that some conditions on the system are required. The reachable canonical form has the property that the parameters of the system are the coefficients of the characteristic equation. It is therefore natural to consider systems on this form when solving the pole placement problem. In the next example we investigate the case when the system is in reachable canonical form.

Example 14 *(System in Reachable Canonical Form).* Consider a system in reachable canonical form, i.e,

$$\frac{dz}{dt} = \tilde{A} z + \tilde{B} u = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \tag{5.12}$$
$$y = \tilde{C} z = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} z$$

The open loop system has the characteristic polynomial

$$D_n(s) = \det \begin{matrix} s+a_1 & a_2 & \dots & a_{n-1} & a_n \\ -1 & s & & 0 & 0 \\ 0 & -1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & -1 & s \end{matrix}$$

Expanding the determinant by the last row we find that the following recursive equation for the determinant.

$$D_n(s) = sD_{n-1}(s) + a_n$$

It follows from this equation that

$$D_n(s) = s^n + a_1 s^{n-1} + \dots + a_{n-1}s + a_n$$

A useful property of the system described by (5.12) is thus that the coefficients of the characteristic polynomial appear in the first row. Since the all elements of the B-matrix except the first row are zero it follows that the state feedback only changes the first row of the A-matrix. It is thus straight forward to see how the closed loop poles are changed by the feedback. Introduce the control law

$$u = -\tilde{L}z + K_r r = -\tilde{k}_1 z_1 - \tilde{k}_2 z_2 - \dots - \tilde{k}_n z_n + K_r r \tag{5.13}$$

The closed loop system then becomes

$$\begin{aligned} \frac{dz}{dt} &= \begin{matrix} -a_1-\tilde{k}_1 & -a_2-\tilde{k}_2 & \dots & -a_{n-1}-\tilde{k}_{n-1} & -a_n-\tilde{k}_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{matrix} \; z + \begin{matrix} K_r \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \; r \\ y &= b_1 \quad b_2 \quad \cdots \quad b_n z \end{aligned} \tag{5.14}$$

The feedback thus changes the elements of the first row of the A matrix, which corresponds to the parameters of the characteristic equation. The closed loop system thus has the characteristic polynomial

$$s^n + (a_l + \tilde{k}_1)s^{n-1} + (a_2 + \tilde{k}_2)s^{n-2} + \dots + (a_{n-1} + \tilde{k}_{n-1})s + a_n + \tilde{k}_n$$

Requiring this polynomial to be equal to the desired closed loop polynomial (5.10) we find that the controller gains should be chosen as

$$\begin{aligned}\tilde{k}_1 &= p_1 - a_1\\ \tilde{k}_2 &= p_2 - a_2\\ &\vdots\\ \tilde{k}_n &= p_n - a_n\end{aligned}$$

This feedback simply replace the parameters a_i in the system (5.14) by p_i. The feedback gain for a system in reachable canonical form is thus

$$\tilde{L} = p_1 - a_1 \quad p_2 - a_2 \quad \cdots \quad p_n - a_n \tag{5.15}$$

To have unit steady state gain the parameter K_r should be chosen as

$$K_r = \frac{a_n + \tilde{k}_n}{b_n} = \frac{p_n}{b_n} \tag{5.16}$$

Notice that it is essential to know the precise values of parameters a_n and b_n in order to obtain the correct steady state gain. The steady state gain is thus obtained by precise calibration. This is very different from obtaining the correct steady state value by integral action, which we shall see in later chapters. We thus find that it is easy to solve the pole placement problem when the system has the structure given by (5.12).

The General Case

To solve the problem in the general case, we simply change coordinates so that the system is in reachable canonical form. Consider the system (5.7). Change the coordinates by a linear transformation

$$z = Tx$$

so that the transformed system is in reachable canonical form (5.12). For such a system the feedback is given by (5.13) where the coefficients are given by (5.15). Transforming back to the original coordinates gives the feedback

$$u = -\tilde{L}z + K_r r = -\tilde{L}Tx + K_r r$$

It now remains to find the transformation. To do this we observe that the reachability matrices have the property

$$\tilde{W}_r = \tilde{B} \quad \tilde{A}\tilde{B} \quad \ldots \quad \tilde{A}^{n-1}\tilde{B} = TB \quad AB \quad \ldots \quad A^{n-1}B = TW_r$$

The transformation matrix is thus given by

$$T = \tilde{W}_r W_r^{-1} \tag{5.17}$$

and the feedback gain can be written as

$$L = \tilde{L}T = \tilde{L}\tilde{W}_r W_r^{-1} \tag{5.18}$$

Notice that the matrix $\tilde{W}_r$ is given by (5.5). The feedforward gain K_r is given by Equation (5.16).

The results obtained can be summarized as follows.

Theorem 4 *(Pole-placement by State Feedback). Consider the system given by Equation* (5.7)

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu\\ y &= Cx\end{aligned}$$

with one input and one output. If the system is reachable there exits a feedback

$$u = -Lx + K_r r$$

that gives a closed loop system with the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \ldots + p_{n-1}s + p_n$$

The feedback gain is given by

$$\begin{aligned}L = \tilde{L}T &= p_1 - a_1 \quad p_2 - a_2 \quad \ldots \quad p_n - a_n \tilde{W}_r W_r^{-1}\\ K_r &= \frac{p_n}{a_n}\end{aligned}$$

where a_i are the coefficients of the characteristic polynomial of the matrix A and the matrices W_r and $\tilde{W}_r$ are given by

$$W_r = B \quad AB \quad \ldots \quad A^{n-1}$$

$$\tilde{W}_r = \begin{matrix} 1 & a_1 & a_2 & \ldots & a_{n-1} \\ 0 & 1 & a_1 & \ldots & a_{n-2} \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 1 \end{matrix}^{-1}$$

Remark 1 *(A mathematical interpretation).* Notice that the pole-placement problem can be formulated abstractly as the following algebraic problem. Given an $n \times n$ matrix A and an $n \times 1$ matrix B, find a $1 \times n$ matrix L such that the matrix $A - BL$ has prescribed eigenvalues.

Computing the Feedback Gain

We have thus obtained a solution to the problem and the feedback has been described by a closed form solution.

For simple problems is is easy to solve the problem by the following simple procedure: Introduce the elements k_i of K as unknown variables. Compute the characteristic polynomial

$$\det(sI - A + BK)$$

Equate coefficients of equal powers of s to the coefficients of the desired characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \ldots + p_{n-1} + p_n$$

This gives a system of linear equations to determine k_i. The equations can always be solved if the system is observable. Example 12 is typical illustrations.

For systems of higher order it is more convenient to use Equation 5.18, this can also be used for numeric computations. However, for large systems this is not sound numerically, because it involves computation of the characteristic polynomial of a matrix and computations of high powers of matrices. Both operations lead to loss of numerical accuracy. For this reason there are other methods that are better numerically. In Matlab the state feedback can be computed by the procedures `acker` or `place`.

Summary

It has been found that the control problem is simple if all states are measured. The most general feedback is a static function from the state space to space of controls. A particularly simple case is when the feedback is restricted to be linear, because it can then be described as a matrix or a vector in the case of systems with only one control variable. A method of determining the feedback gain in such a way that the closed loop system has prescribed poles has been given. This can always be done if the system is reachable.

5.4 Observers

Advanced

In Section 5.3 it was shown that the pole it was possible to find a feedback that gives desired closed loop poles provided that the system is reachable and that all states were measured. It is highly unrealistic to assume that all states are measured. In this section we will investigate how the state can be estimated by using the mathematical model and a few measurements. It will be shown that the computation of the states can be done by dynamical systems. Such systems will be called observers.

Consider a system described by

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu \\ y &= Cx\end{aligned} \tag{5.19}$$

where x is the state, u the input, and y the measured output. The problem of determining the state of the system from its inputs and outputs will be considered. It will be assumed that there is only one measured signal, i.e. that the signal y is a scalar and that C is a vector.

Observability

When discussing reachability we neglected the output and focused on the state. We will now discuss a related problem where we will neglect the input and instead focus on the output. Consider the system

$$\begin{aligned}\frac{dx}{dt} &= Ax \\ y &= Cx\end{aligned} \tag{5.20}$$

We will now investigate if it is possible to determine the state from observations of the output. This is clearly a problem of significant practical interest, because it will tell if the sensors are sufficient.

The output itself gives the projection of the state on vectors that are rows of the matrix C. The problem can clearly be solved if the matrix C is invertible. If the matrix is not invertible we can take derivatives of the output to obtain.

$$\frac{dy}{dt} = C\frac{dx}{dt} = CAx$$

From the derivative of the output we thus get the projections of the state on vectors which are rows of the matrix CA. Proceeding in this way we get

$$\begin{pmatrix} y \\ \frac{dy}{dt} \\ \frac{d^2y}{dt^2} \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}} \end{pmatrix} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} x$$

We thus find that the state can be determined if the matrix

$$W_o = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} \tag{5.21}$$

has n independent rows. Notice that because of the Cayley-Hamilton equation it is not worth while to continue and take derivatives of order higher than $n-1$. The matrix W_o is called the observability matrix. A system is called observable if the observability matrix has full rank. We illustrate with an example.

Example 15 *(Observability of the Inverted Pendulum).* The linearized model of inverted pendulum around the upright position is described by (**??**). The matrices A and C are

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

The observability matrix is

$$W_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which has full rank. It is thus possible to compute the state from a measurement of the angle.

A Non-observable System

It is useful to have an understanding of the mechanisms that make a system unobservable. Such a system is shown in Figure 5.3. Next we will consider

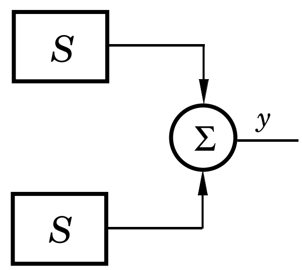

Figure 5.3: A non-observable system.

the system in *observable canonical form*, i.e.

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 \dots & 0 \end{pmatrix} z + Du$$

A straight forward but tedious calculation shows that the inverse of the observability matrix has a simple form. It is given by

$$W_o^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ a_1 & 1 & 0 & \dots & 0 \\ a_2 & a_1 & 1 & \dots & 0 \\ \vdots & & & & \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & 1 \end{pmatrix}$$

This matrix is always invertible. The system is composed of two identical systems whose outputs are added. It seems intuitively clear that it is not possible to deduce the states from the output. This can also be seen formally.

Coordinate Changes

It is interesting to investigate how the observability matrix transforms when the coordinates are changed. Consider the system in equation (5.20). Assume that the coordinates are changed to $z = Tx$. It follows from linear algebra that the dynamics matrix and the output matrix are given by

$$\begin{aligned}\tilde{A} &= TAT^{-1}\\ \tilde{C} &= CT^{-1}.\end{aligned}$$

The observability matrix for the transformed system then becomes

$$\tilde{W}_o = \begin{pmatrix}\tilde{C}\\ \tilde{C}\tilde{A}\\ \tilde{C}\tilde{A}^2\\ \vdots\\ \tilde{C}\tilde{A}^{n-1}\end{pmatrix}$$

We have

$$\begin{aligned}\tilde{C}\tilde{A} &= CT^{-1}TAT^{-1} = CAT^{-1}\\ \tilde{C}\tilde{A}^2 &= CT^{-1}(TAT^{-1})^2 = CT^{-1}TAT^{-1}TAT^{-1} = CA^2T^{-1}\\ &\vdots\\ \tilde{C}\tilde{A}^n &= CA^nT^{-1}\end{aligned}$$

and we find that the observability matrix for the transformed system has the property

$$\tilde{W}_o = \begin{pmatrix}\tilde{C}\\ \tilde{C}\tilde{A}\\ \tilde{C}\tilde{A}^2\\ \vdots\\ \tilde{C}\tilde{A}^{n-1}\end{pmatrix} T^{-1} = W_oT^{-1} \tag{5.22}$$

This formula is very useful for finding the transformation matrix T.

Observers Based on Differentiation

An observer based on differentiation will first be given. The construction is an extension of the derivation of the criterion for observability given above.

First observe that the output equation

$$y = Cx$$

gives the projection of the state on the vector C. Differentiation of this equation gives

$$\frac{dy}{dt} = C\frac{dx}{dt} = CAx + CBu$$

The derivative of the output together with CBu thus gives the projection of the state vector on the vector CA. Proceeding in this way and taking higher derivatives give the projections of the state vector on the vectors $C, CA, \ldots, CA^{n-1}$. If these vectors are linearly independent, the projections of the state on n linearly independent vectors are obtained and the state can thus be determined. Carrying out the details, we get

$$
\begin{aligned}
y &= Cx \\
\frac{dy}{dt} &= Cdxdt = CAx + CBu \\
\frac{d^2y}{dt^2} &= CA\frac{dx}{dt} + CB\frac{du}{dt} = CA^2x + CABu + CB\frac{du}{dt} \\
&\vdots \\
\frac{d^{n-1}y}{dt^{n-1}} &= CA^{n-1}x + CA^{n-2}Bu + CA^{n-3}B\frac{du}{dt} + \ldots + CB\frac{d^{n-2}u}{dt^{n-2}}
\end{aligned}
$$

This equation can be written in matrix form as

$$
\begin{matrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{matrix} \; x = \begin{matrix} y \\ \frac{dy}{dt} - CBu \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}} - CA^{n-2}Bu - CA^{n-3}B\frac{du}{dt} - \ldots - CB\frac{d^{n-2}u}{dt^{n-2}} \end{matrix}
$$

Notice that the matrix on the left-hand side is the observability matrix W_o. If the system is observable, the equation can be solved to give

$$
x = W_o^{-1} \begin{matrix} y \\ \frac{dy}{dt} \\ \vdots \\ \frac{d^{n-1}}{dt^{n-1}} \end{matrix} - W_o^{-1} \begin{matrix} 0 & 0 & \cdots & 0 \\ CB & 0 & \cdots & 0 \\ \vdots & & & \\ CA^{n-2}B & CA^{n-3}B & \cdots & CB \end{matrix} \begin{matrix} u \\ \frac{du}{dt} \\ \vdots \\ \frac{d^{n-2}}{dt^{n-2}} \end{matrix} \tag{5.23}
$$

This is an exact expression for the state. The state is obtained by differentiating inputs and outputs. Notice that it has been derived under the assumption that there is no measurement noise. Differentiation can give very large errors when there is measurement noise and the method is therefore not very practical particularly when derivatives of high order appear.

Using a Dynamical System to Observe the State

For a system governed by Equation (5.19), it can be attempted to determine the state simply by simulating the equations with the correct input. An estimate of the state is then given by

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu \tag{5.24}$$

To find the properties of this estimate, introduce the estimation error

$$\tilde{x} = x - \hat{x}$$

It follows from (5.19) and (5.24) that

$$\frac{d\tilde{x}}{dt} = A\tilde{x}$$

If matrix A has all its eigenvalues in the left half plane, the error $\tilde{x}$ will thus go to zero. Equation (5.24) is thus a dynamical system whose output converges to the state of the system (5.19).

The observer given by (5.24) uses only the process input u, the measured signal does not appear in the equation. It must also be required that the system is stable. We will therefore attempt to modify the observer so that the output is used and that it will work for unstable systems. Consider the following

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{5.25}$$

observer. This can be considered as a generalization of (5.24). Feedback from the measured output is provided by adding the term $L(y-C\hat{x})$. Notice that $C\hat{x} = \hat{y}$ is the output that is predicted by the observer. To investigate the observer (5.25), form the error

$$\tilde{x} = x - \hat{x}$$

It follows from (5.19) and (5.25) that

$$\frac{d\tilde{x}}{dt} = (A - LC)\tilde{x}$$

If the matrix L can be chosen in such a way that the matrix $A - LC$ has eigenvalues with negative real parts, error $\tilde{x}$ will go to zero. The convergence rate is determined by an appropriate selection of the eigenvalues.

The problem of determining the matrix L such that $A - LC$ has prescribed eigenvalues is very similar to the pole placement problem that was solved above. In fact, if we observe that the eigenvalues of the matrix and its transpose are the same, we find that could determine L such that $A^T - C^T L^T$

has given eigenvalues. First we notice that the problem can be solved if the matrix

$$C^T \quad A^T C^T \quad \ldots \quad A^{(n-1)T} C^T$$

is invertible. Notice that this matrix is the transpose of the observability matrix for the system (5.19).

$$W_o = \begin{matrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{matrix}$$

of the system. Assume it is desired that the characteristic polynomial of the matrix $A - LC$ is

$$p(s) = s^n + p_1 s^{n-1} + \ldots + p_n$$

It follows from Remark 1 of Theorem 4 that the solution is given by

$$L^T = p_1 - a_1 \quad p_2 - a_2 \quad \ldots \quad p_n - a_n \tilde{W}_o^T W_o^{-T}$$

where W_o is the observability matrix and $\tilde{W}_o$ is the observability matrix of the system of the system

$$\frac{dz}{dt} = \begin{matrix} -a_1 & 1 & 0 & \ldots & 0 \\ -a_2 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & \ldots & 1 \\ -a_n & 0 & 0 & \ldots & 0 \end{matrix} z + \begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{matrix} u$$

$$y = 1 \quad 0 \quad 0 \quad \ldots \quad 0$$

which is the observable canonical form of the system (5.19). Transposing the formula for K we obtain

$$K = W_o^{-1} \tilde{W}_o \begin{matrix} p_1 - a_1 \\ p_2 - a_2 \\ \vdots \\ p_n - a_n \end{matrix}$$

The result is summarized by the following theorem.

Theorem 5 *(Observer design by pole placement). Consider the system given by*

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$

where output y is a scalar. Assume that the system is observable. The dynamical system

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

with K chosen as

$$L = W_o^{-1}\tilde{W}_o \begin{matrix} p_1 - a_1 \\ p_2 - a_2 \\ \vdots \\ p_n - a_n \end{matrix} \tag{5.26}$$

where the matrices W_o and $\tilde{W}_o$ are given by

$$W_o = \begin{matrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{matrix}, \quad \tilde{W}_o^{-1} = \begin{matrix} 1 & 0 & 0 & \dots & 0 \\ a_1 & 1 & 0 & \dots & 0 \\ a_2 & a_1 & 1 & \dots & 0 \\ \vdots & & & & \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & 1 \end{matrix}$$

Then the observer error $\tilde{x} = x - \hat{x}$ is governed by a differential equation having the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \ldots + p_n$$

Remark 2. The dynamical system (5.25) is called an observer for (the states of the) system (5.19) because it will generate an approximation of the states of the system from its inputs and outputs.

Remark 3. The theorem can be derived by transforming the system to observable canonical form and solving the problem for a system in this form.

Remark 4. Notice that we have given two observers, one based on pure differentiation (5.23) and another described by the differential equation (5.25). There are also other forms of observers.

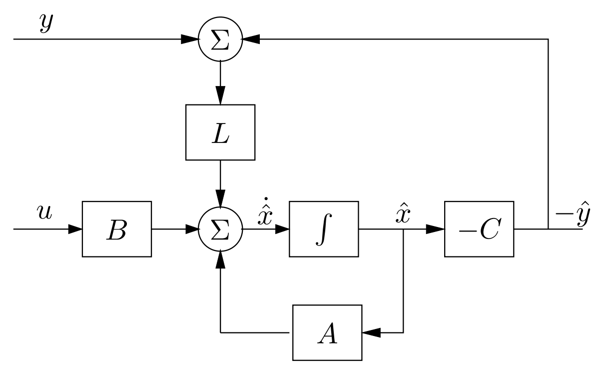

Figure 5.4: Block diagram of the observer. Notice that the observer contains a copy of the process.

Interpretation of the Observer

The observer is a dynamical system whose inputs are process input u and process output y. The rate of change of the estimate is composed of two terms. One term $A\hat{x} + Bu$ is the rate of change computed from the model with $\hat{x}$ substituted for x. The other term $K(y - \hat{y})$ is proportional to the difference $e = y - \hat{y}$ between measured output y and its estimate $\hat{y} = C\hat{x}$. The estimator gain L is a matrix that tells how the error e is weighted and distributed among the states. The observer thus combines measurements with a dynamical model of the system. A block diagram of the observer is shown in Figure 5.4.

Duality

Notice the similarity between the problems of finding a state feedback and finding the observer. The key is that both of these problems are equivalent to the same algebraic problem. In pole placement it is attempted to find K so that $A - BK$ has given eigenvalues. For the observer design it is instead attempted to find L so that $A - LC$ has given eigenvalues. The following equivalence can be established between the problems

$$
\begin{aligned}
A &\leftrightarrow A^T \\
B &\leftrightarrow C^T \\
K &\leftrightarrow L^T \\
W_r &\leftrightarrow W_o^T
\end{aligned}
$$

The similarity between design of state feedback and observers also means that the same computer code can be used for both problems.

Computing the Observer Gain

The observer gain can be computed in several different ways. For simple problems it is convenient to introduce the elements of L as unknown parameters, determine the characteristic polynomial of the observer $\det(A - LC)$ and identify it with the desired characteristic polynomial. Another alternative is to use the fact that the observer gain can be obtained by inspection if the system is in observable canonical form. In the general case the observer gain is then obtained by transformation to the canonical form. There are also reliable numerical algorithms. They are identical to the algorithms for computing the state feedback. The procedures are illustrated by a few examples.

Example 16 *(The Double Integrator).* The double integrator is described by

$$\frac{dx}{dt} = \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} x + \begin{matrix} 0 \\ 1 \end{matrix} u$$
$$y = \begin{matrix} 1 & 0 \end{matrix}$$

The observability matrix is

$$W_o = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

i.e. the identity matrix. The system is thus observable and the problem can be solved. We have

$$A - LC = \begin{matrix} -l_1 & 1 \\ -l_2 & 0 \end{matrix}$$

It has the characteristic polynomial

$$\det A - LC = \det \begin{matrix} s + l_1 & -1 \\ -l_2 & s \end{matrix} = s^2 + l_1 s + l_2$$

Assume that it is desired to have an observer with the characteristic polynomial

$$s^2 + p_1 s + p_2 = s^2 + 2\zeta\omega s + \omega^2$$

The observer gains should be chosen as

$$l_1 = p_1 = 2\zeta\omega$$
$$l_2 = p_2 = \omega^2$$

The observer is then

$$\frac{d\hat{x}}{dt} = \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \hat{x} + \begin{matrix} 0 \\ 1 \end{matrix} u + \begin{matrix} l_1 \\ l_2 \end{matrix} (y - \hat{x}_1)$$

5.5 Output Feedback

In this section we will consider the same system as in the previous sections, i.e. the nth order system described by

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{5.27}$$

where only the output is measured. As before it will be assumed that u and y are scalars. It is also assumed that the system is reachable and observable. In Section 5.3 we had found a feedback

$$u = -Kx + K_r r$$

for the case that all states could be measured and in Section 5.4 we have presented developed an observer that can generate estimates of the state $\hat{x}$ based on inputs and outputs. In this section we will combine the ideas of these sections to find an feedback which gives desired closed loop poles for systems where only outputs are available for feedback.

If all states are not measurable, it seems reasonable to try the feedback

$$u = -K\hat{x} + K_r r \tag{5.28}$$

where $\hat{x}$ is the output of an observer of the state (5.25) ,i.e.

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{5.29}$$

Since the system (5.27) and the observer (5.29) both are of order n, the closed loop system is thus of order $2n$. The states of the system are x and $\hat{x}$. The evolution of the states is described by equations (5.27), (5.28)(5.29). To analyze the closed loop system, the state variable $\hat{x}$ is replace by

$$\tilde{x} = x - \hat{x} \tag{5.30}$$

Subtraction of (5.27) from (5.27) gives

$$\frac{d\tilde{x}}{dt} = Ax - A\hat{x} - K(y - C\hat{x}) = A\tilde{x} - KC\tilde{x} = (A - LC)\tilde{x}$$

Introducing u from (5.28) into this equation and using (5.30) to eliminate $\hat{x}$ gives

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu = Ax - BK\hat{x} + BK_r r = Ax - BK(x - \tilde{x}) + BK_r r \\ &= (A - BK)x + BK\tilde{x} + BK_r r\end{aligned}$$

The closed loop system is thus governed by

$$\frac{d}{dt}\begin{pmatrix} x \\ \tilde{x}\end{pmatrix} = \begin{pmatrix} A - BK & BK \\ 0 & A - LC\end{pmatrix}\begin{pmatrix} x \\ \tilde{x}\end{pmatrix} + \begin{pmatrix} BK_r \\ 0\end{pmatrix} r \tag{5.31}$$

Since the matrix on the right-hand side is block diagonal, we find that the characteristic polynomial of the closed loop system is

$$\det(sI - A + BK)\det(sI - A + LC)$$

This polynomial is a product of two terms, where the first is the characteristic polynomial of the closed loop system obtained with state feedback and the other is the characteristic polynomial of the observer error. The feedback (5.28) that was motivated heuristically thus provides a very neat solution to the pole placement problem. The result is summarized as follows.

Theorem 6 *(Pole placement by output feedback). Consider the system*

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu \\ y &= Cx\end{aligned}$$

The controller described by

$$\begin{aligned}u &= -K\hat{x} + K_r r \\ \frac{d\hat{x}}{dt} &= A\hat{x} + Bu + L(y - C\hat{x})\end{aligned}$$

gives a closed loop system with the characteristic polynomial

$$\det(sI - A + BK)\det(sI - A + LC)$$

This polynomial can be assigned arbitrary roots if the system is observable and reachable.

Remark 5. Notice that the characteristic polynomial is of order $2n$ and that it can naturally be separated into two factors, one $\det(sI - A + BK)$ associated with the state feedback and the other $\det(sI - A + LC)$ with the observer.

Remark 6. The controller has a strong intuitive appeal. It can be thought of as composed of two parts, one state feedback and one observer. The feedback gain L can be computed as if all state variables can be measured.

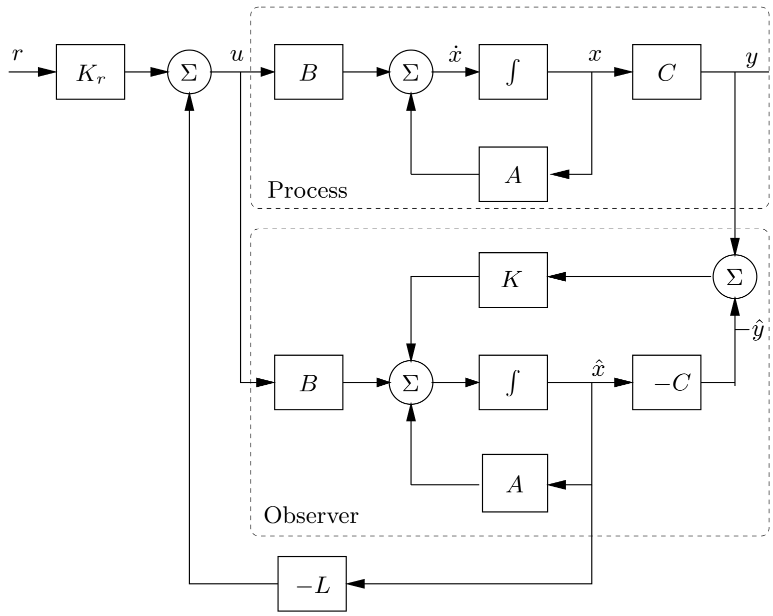

Figure 5.5: Block diagram of a controller which combines state feedback with an observer.

The Internal Model Principle

A block diagram of the controller is shown in Figure 5.5. Notice that the controller contains a dynamical model of the plant. This is called the internal model principle. Notice that the dynamics of the controller is due to the observer. The controller can be viewed as a dynamical system with input y and output u.

$$\begin{aligned}\frac{d\hat{x}}{dt} &= (A - BK - LC)\hat{x} + Ky \\ u &= -L\hat{x} + K_r r\end{aligned}$$

The controller has the transfer function

$$C(s) = L[sI - A + BK + LC]^{-1}K \tag{5.32}$$

5.6 Summary

In this chapter we have presented a systematic method for design of a controller. The controller has an interesting structure, it can be thought of as composed of three subsystems: a system that generates the desired output and a feedforward signals from the reference value, an estimator and a feedback from estimated states. This structure has the property that the response to reference signals can be decoupled from the response to disturbances. The details are carried out only for systems with one input and one output but it turns out that the structure of the controller is the same for systems with many inputs and many outputs. The equations for the controller have the same form, the only difference is that the feedback gain L and the observer gain K are matrices instead of vectors for the single-input single-output case. There are also many other design methods that give controllers with the same structure but the gains K and L are computed differently. The analysis also gives an interesting interpretation of integral action as a disturbance estimator. This admits generalizations to many other types of disturbances.