State-Dependent Systems: Markov Analysis

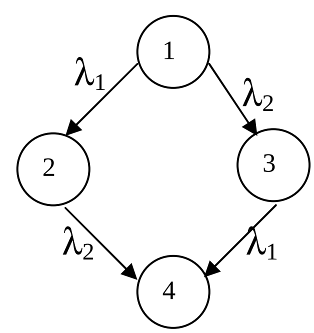
Markov analysis

Markov analysis looks at a system being in one of several states, for example all components composing the system are operating. Another possible state is that in which one component has failed but the others component continue to operate.

- The main assumption in a Markov analysis is that the probability that the system will under goes transition from one state to another depends only on the current state of the system and not on states of the system that the system may have experienced.
- In other words, transition probability is not dependent on the history of the system
- This, memorylessness, property is similar to exponential distribution

Two components system

State	Component 1	Component 2
1	Operating	Operating
2	Failed	Operating
3	Operating	Failed
4	Failed	Failed



- If the two components are parallel (redundant), only state 4 results in system failure. Reliability of the system would be:

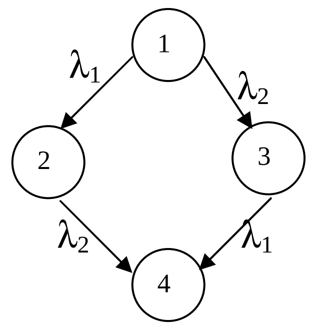
  $R_s(t) = P_1(t)+ P_2(t)+ P_3(t)$; Where is P_i is probability of operation

- If two components are in series, then states 2, 3, and 4 would result is failure. Reliability of the system would be:

  $R_s(t) = P_1(t)$; Where is P_1 is probability of operation

Considering constant failure rate, the system may be represented as shown in above figures. From the above figure, we can drive

$$\frac{dP_1(t)}{dt} = -(\lambda_1 + \lambda_2)P_1(t)$$
$$\frac{dP_2(t)}{dt} = \lambda_1 P_1(t) - \lambda_2 P_2(t)$$
$$\frac{dP_3(t)}{dt} = \lambda_2 P_1(t) - \lambda_1 P_3(t)$$



Solving these simultaneous differential equations:

$$P_1(t) = e^{-(\lambda_1+\lambda_2)t}$$
$$P_2(t) = e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}$$
$$P_3(t) = e^{-\lambda_1 t} - e^{-(\lambda_1+\lambda_2)t}$$
$$P_4(t) = 1 - P_1(t) - P_2(t) - P_3(t)$$

Reliability of series system

$$R_s(t) = P_1(t) = e^{-(\lambda_1+\lambda_2)t}$$

Reliability of a parallel system

$$R_p(t) = P_1(t) + P_2(t) + P_3(t) = e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}$$

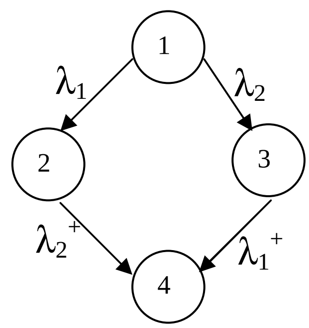
Load sharing system

- Two component are in parallel (redundancy) but have a dependency
- If one component fails, the failure rate of the other component increase as a result of additional load placed on it
- The defined states remain the same, however, λ_1^+, λ_2^+ are increased failures rates of component 1 and 2.

$$\frac{dP_1(t)}{dt} = -(\lambda_1 + \lambda_2)P_1(t)$$

$$\frac{dP_2(t)}{dt} = \lambda_1 P_1(t) - \lambda_2^+ P_2(t)$$

$$\frac{dP_3(t)}{dt} = \lambda_2 P_1(t) - \lambda_1^+ P_3(t)$$



Solving these simultaneous differential equations:

$$P_1(t) = e^{-(\lambda_1+\lambda_2)t}$$

$$P_2(t) = \frac{\lambda_1}{\lambda_1 + \lambda_2 - \lambda_2^+}\left[e^{-\lambda_2^+ t} - e^{-(\lambda_1+\lambda_2)t}\right]$$

$$P_3(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2 - \lambda_1^+}\left[e^{-\lambda_1^+ t} - e^{-(\lambda_1+\lambda_2)t}\right]$$

$$P_4(t) = 1 - P_1(t) - P_2(t) - P_3(t)$$

Reliability of the system

$R(t) = P_1(t) + P_2(t) + P_3(t)$

If $\lambda_1^+ = \lambda_2^+ = \lambda^+$ and $\lambda_1 = \lambda_2 = \lambda$

$$R(t) = e^{-2\lambda t} + \frac{2\lambda}{2\lambda - \lambda^+}\left[e^{-\lambda^+ t} - e^{-2\lambda t}\right]$$

$$MTTF = \int_0^\infty R(t)dt = \frac{1}{2\lambda} + \frac{2\lambda}{2\lambda - \lambda^+}\left[\frac{1}{\lambda^+} - \frac{1}{2\lambda}\right]$$

Standby system

The standby system differs from the active redundant system in a way that standby system will have no failure or reduce failure while in standby mode.

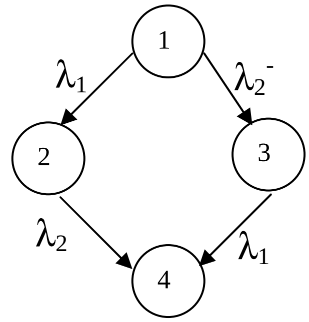
- Once standby component become active it may experience the same failure rate as the online system (for identical components) or different failure rate.
- In other words, failure rate of standby unit depends on the state of the primary unit.
- Depending on the probability of failure during switching to a standby unit, these system are generally much more reliable than online redundant system.

Figure below represents the system, where state 3 represents a failure of the standby system with λ_2^- as failure rate.

$$\frac{dP_1(t)}{dt} = -(\lambda_1 + \lambda_2^-)P_1(t)$$

$$\frac{dP_2(t)}{dt} = \lambda_1 P_1(t) - \lambda_2 P_2(t)$$

$$\frac{dP_3(t)}{dt} = \lambda_2^- P_1(t) - \lambda_1 P_3(t)$$



Solving these simultaneous differential equations:

$$P_1(t) = e^{-(\lambda_1 + \lambda_2^-)t}$$

$$P_2(t) = \frac{\lambda_1}{\lambda_1 + \lambda_2^- - \lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2^-)t}\right]$$

$$P_3(t) = e^{-\lambda_1 t} - e^{-(\lambda_1 + \lambda_2^-)t}$$

$$P_4(t) = 1 - P_1(t) - P_2(t) - P_3(t)$$

Reliability of the system

R(t) = P_1(t)+ P_2(t)+ P_3(t)

$$R(t) = e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_1 + \lambda_2^- - \lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2^-)t}\right]$$

$$MTTF = \int_0^\infty R(t)dt = \frac{1}{\lambda_1} + \frac{\lambda_1}{\lambda_2(\lambda_1 + \lambda_2^-)}$$

Identical Standby Units

Considering there are k identical units of which one is on-line and remaining are standby, when online unit fails, the first standby is placed on and so on. The time in which k^th failure will be observed is the sum of k identical and independent exponential distribution. The reliability of k identical system will be:

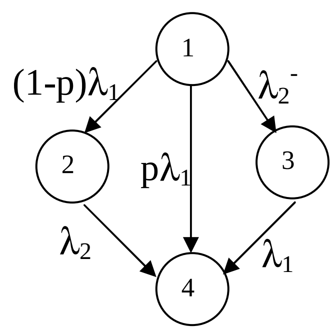
$$R_k(t) = e^{-\lambda t} \sum_{i=0}^{k-1} \frac{(\lambda t)^i}{i!}$$

$$MTTF = \frac{k}{\lambda}$$

Standby system with Switching Failures

- The standby system may fail during switching from standby mode to online.
- Let p is the probability of failure of switching device on demand that switches the standby system to operation.

Below figure shows the state of the system.



The resulting equations are:

$$\frac{dP_1(t)}{dt} = -\left[(1-p)\lambda_1 + p\lambda_1 + \lambda_2^-\right]P_1(t)$$

$$= -(\lambda_1 + \lambda_2^-)P_1(t)$$

$$\frac{dP_2(t)}{dt} = (1-p)\lambda_1 P_1(t) - \lambda_2 P_2(t)$$

Solving these simultaneous differential equations

$$P_1(t) = e^{-(\lambda_1+\bar{\lambda}_2)t}$$

$$P_2(t) = \frac{(1-p)\lambda_1}{\lambda_1+\bar{\lambda}_2-\lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1+\bar{\lambda}_2)t}\right]$$

$$P_3(t) = e^{-\lambda_1 t} - e^{-(\lambda_1+\bar{\lambda}_2)t}$$

$$P_4(t) = 1 - P_1(t) - P_2(t) - P_3(t)$$

Reliability of the system

$R(t) = P_1(t) + P_2(t) + P_3(t)$

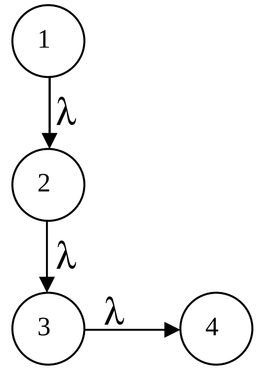
$$R(t) = e^{-\lambda_1 t} + \frac{(1-p)\lambda_1}{\lambda_1+\bar{\lambda}_2-\lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1+\bar{\lambda}_2)t}\right]$$

Three components Standby system

Considering a system with one active and two standby units.

Considering no units fails while in standby and all the three units have same constant failure rate when on-line.

State	Unit 1	Unit 2	Unit 3
1	Operating	Standby	Standby
2	Failed	Operating	Standby
3	Failed	Failed	Operating
4	Failed	Failed	Failed



$$\frac{dP_1(t)}{dt} = -\lambda P_1(t)$$

$$\frac{dP_2(t)}{dt} = \lambda P_1(t) - \lambda P_2(t)$$

$$\frac{dP_3(t)}{dt} = \lambda P_2(t) - \lambda P_3(t)$$

With initial conditions $P_1(t) = 0$, $P_2(t) = 0$, and $P_3(t) = 0$, Solving these differential equations:

$$P_1(t) = e^{-\lambda t}$$

$$P_2(t) = \lambda t e^{-\lambda t}$$

$$P_3(t) = \frac{(\lambda t)^2 e^{-\lambda t}}{2}$$

Reliability of the system

$R(t) = P_1(t) + P_2(t) + P_3(t)$

$$R(t) = e^{-\lambda_1 t}\left[1 + \lambda t + \frac{(\lambda t)^2}{2}\right]$$

$$MTTF = \int_0^\infty R(t)dt = \frac{3}{\lambda}$$

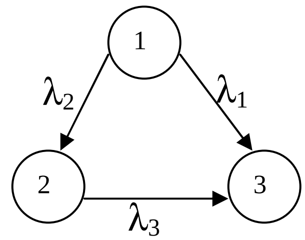
- As may be evident from the results that it is the same case as discussed in the identical standby unit section.
- System having three identical units with one online and two standby will have MTTF three times higher than single unit system

Degraded System

In many situation system may continue to operate in a degraded mode (less than required level) following certain types of failures

Such situation may be represented by three states:

State 1: Fully operational, State 2: degraded state; State 3: Failed state



$$\frac{dP_1(t)}{dt} = -(\lambda_1 + \lambda_2)P_1(t)$$

$$\frac{dP_2(t)}{dt} = \lambda_2 P_1(t) - \lambda_3 P_2(t)$$

The solution is

$$P_1(t) = e^{-(\lambda_1+\lambda_2)t}$$

$$P_2(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2 - \lambda_3}\left[e^{-\lambda_3 t} - e^{-(\lambda_1+\lambda_2)t}\right]$$

$$P_3(t) = 1 - P_1(t) - P_2(t)$$

Reliability of the system

$R(t) = P_1(t) + P_2(t)$

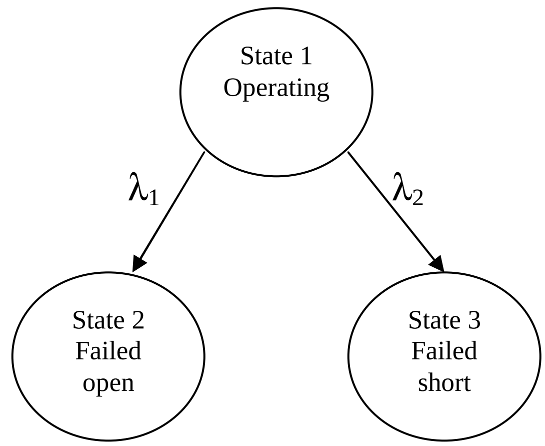
$$R(t) = e^{-(\lambda_1+\lambda_2)t} + \frac{\lambda_2}{\lambda_1 + \lambda_2 - \lambda_3}\left[e^{-\lambda_3 t} - e^{-(\lambda_1+\lambda_2)t}\right]$$

$$MTTF = \int_0^\infty R(t)dt = \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2 - \lambda_3}\left[\frac{1}{\lambda_3} - \frac{1}{\lambda_1 + \lambda_2}\right]$$

Three State Devices

Components having three states (operating stage, failed open, and failed short) may be analyzed using Markov analysis

The differential equations are:



$$\frac{dP_1(t)}{dt} = -(\lambda_1 + \lambda_2)P_1(t)$$

$$\frac{dP_2(t)}{dt} = \lambda_1 P_1(t)$$

$$\frac{dP_3(t)}{dt} = \lambda_2 P_1(t)$$

The solution is

$$P_1(t) = e^{-(\lambda_1+\lambda_2)t}$$

$$P_2(t) = \frac{\lambda_1}{\lambda_1 + \lambda_2}\left[1 - e^{-(\lambda_1+\lambda_2)t}\right]$$

$$P_3(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2}\left[1 - e^{-(\lambda_1+\lambda_2)t}\right]$$

Reliability of the system

$$R(t) = P_1(t) = e^{-(\lambda_1+\lambda_2)t}$$

$$MTTF = \int_0^{\infty} R(t)dt = \frac{1}{\lambda_1 + \lambda_2}$$