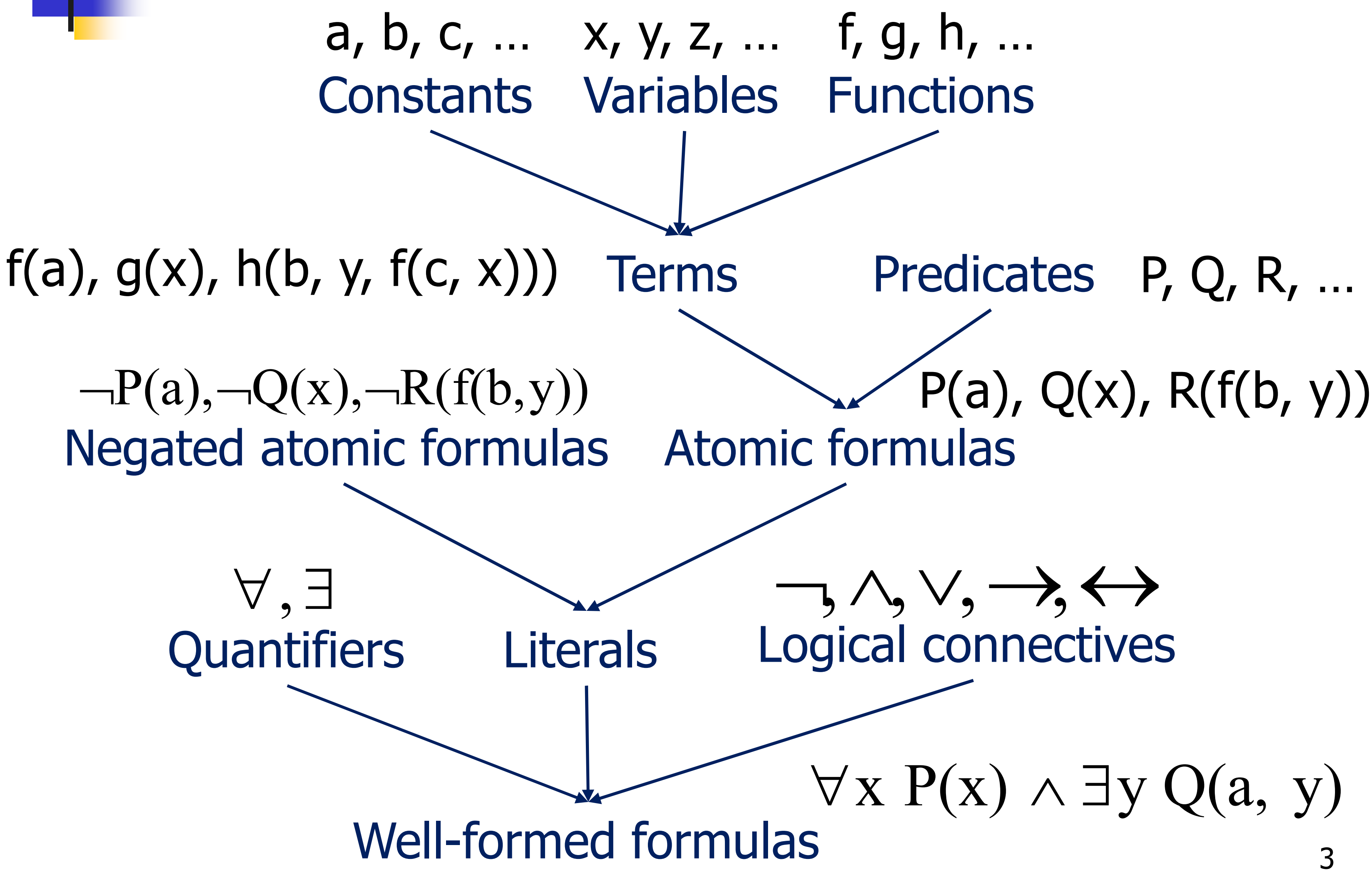
KRR Lecture 3

First Order Predicate Logic

Outline

- Syntax and semantics of FOPL
- Logical consequence
- Proof theory
- Normal forms
- Resolution

FOPL Syntax



a, b, c, …
Constants
x, y, z, …
Variables
f, g, h, …
Functions
f(a), g(x), h(b, y, f(c, x)))
Terms
Predicates
P, Q, R, …
$\neg P(a), \neg Q(x), \neg R(f(b, y))$
Negated atomic formulas
Atomic formulas
P(a), Q(x), R(f(b, y))
$\forall, \exists$
Quantifiers
Literals
$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
Logical connectives
$\forall x\ P(x) \wedge \exists y\ Q(a, y)$
Well-formed formulas

FOPL Syntax – example

- Horses are faster than dogs.
- Some greyhounds are faster than any rabbit.
- Greyhounds are dogs.
- “Faster than” is a transitive relation.
- Bunny is a rabbit.
- Harry is a horse.

- Harry is faster than Bunny.

FOPL Syntax – example (cont.)

$$\forall x \forall y (Horse(x) \land Dog(y) \rightarrow Faster(x, y))$$

$$\exists x (Greyhound(x) \land \forall y (Rabbit(y) \rightarrow Faster(x, y)))$$

$$\forall x (Greyhound(x) \rightarrow Dog(x))$$

$$\forall x \forall y \forall z (Faster(x, y) \land Faster(y, z) \rightarrow Faster(x, z))$$

$$Rabbit(Bunny)$$

$$Horse(Harry)$$

$$Faster(Harry, Bunny)$$

FOPL Semantics

- The interpretation of a FOPL formula consists in choosing a non-empty domain of values D and by associating a value to each constant, function and predicate, as follows:
  - Every constant is associated to a value from D
  - Every function of arity n is associated to a function $D^n \longrightarrow D$
  - Every predicate of arity n is associated to a function $D^n \longrightarrow \{ T, F \}$

FOPL Semantics – example

- D = {a1, a2, a3} – domain of values.
- Harry = a1, Bunny = a3

x	**a1**	**a2**	**a3**
Rabbit(x)	F	F	T
Horse(x)	T	F	F
Dog(x)	F	T	F
Greyhound(x)	F	T	F

- Faster(a1,a2) = Faster(a2,a3) = T
- Faster(x,y) = F, for all other values.

FOPL Semantics – example

$$\forall x \forall y (Horse(x) \wedge Dog(y) \rightarrow Faster(x, y)) = T$$

$$\exists x (Greyhound(x) \wedge \forall y (Rabbit(y) \rightarrow Faster(x, y))) = T$$

$$\forall x (Greyhound(x) \rightarrow Dog(x)) = T$$

$$\forall x \forall y \forall z (Faster(x, y) \wedge Faster(y, z) \rightarrow Faster(x, z)) = F$$

$$Rabbit(Bunny) = T$$

$$Horse(Harry) = T$$

$$Faster(Harry, Bunny) = T$$

Types of formulas

- If a formula is true for a given interpretation (model) M, we say that the model **satisfies** the formula: $M \models \phi$
- 3 types of formulas:
  - **valid / tautologies** = satisfied by all possible models
  - **contingent** = satisfied by some, but not all models
  - **unsatisfiable / inconsistent** = satisfied by no model
- **Satisfiable** = valid or contingent
- **Falsifiable** = contingent or inconsistent

Types of formulas – example

$$\forall x(P(x) \vee \neg P(x))$$

$$\exists x(P(x) \wedge \neg P(x))$$

$$\forall x(P(x) \rightarrow \exists y(Q(y) \wedge R(x, y)))$$

Types of formulas – example

$$\forall x(P(x) \vee \neg P(x)) = \text{valid}$$

$$\exists x(P(x) \wedge \neg P(x)) = \text{inconsistent}$$

$$\forall x(P(x) \rightarrow \exists y(Q(y) \wedge R(x, y))) = \text{contingent}$$

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Logical consequence

- A formula C (conclusion) is a logical consequence of a set of formulas P (premises) iff all models that satisfy all formulas in P also satisfy C

$$P \models C$$

Logical consequence – example

$$\frac{\begin{array}{l}\forall x(Greyhound(x) \rightarrow Dog(x)) \\ Greyhound(Arrow)\end{array}}{Dog(Arrow)}$$

Equivalent formulations

- **Theorem:** C is a logical consequence of {P1, …, Pn} iff the formula

$$P_1 \wedge \ldots \wedge P_n \rightarrow C$$

is valid

- **Theorem:** C is a logical consequence of {P1, …, Pn} iff the formula

$$P_1 \wedge \ldots \wedge P_n \wedge \neg C$$

is inconsistent

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Inference rules

- Valid **schemas** that allow us to (syntactically) infer a conclusion based on a set of premises.
- Modus Ponens (MP)

$$\frac{\begin{array}{l}\Phi \to \Psi \\ \Phi\end{array}}{\Psi} \qquad\qquad \frac{\begin{array}{l}\forall x(P(x) \to Q(x)) \\ P(a)\end{array}}{Q(a)}$$

Inference rules

- Universal instantiation (UInst)

$$\frac{\forall x \Phi(x)}{\Phi(a)}$$

- Existential instantiation (EInst)

$$\frac{\exists x \Phi(x)}{\Phi(a)}$$

- Restrictions

Inference rules

- Conjunction elimination (CElim)

$$\frac{\Phi \wedge \Psi}{\Phi}$$

- Conjunction introduction (CIntr)

$$\frac{\begin{array}{l}\Phi \\ \Psi\end{array}}{\Phi \wedge \Psi}$$

Proof theory

- A conclusion C is provable from a set of premises P if C can be deduced from P using the inference rules available in the proof system

$$P \mid\!- C$$

- **Sound** = cannot prove false conclusions from true premises
- **Complete** = can prove any logical consequence of the premises

Proof example

C. $Faster(Harry, Bunny)$

1. $\forall x \forall y (Horse(x) \wedge Dog(y) \rightarrow Faster(x, y))$

2. $\exists x (Greyhound(x) \wedge \forall y (Rabbit(y) \rightarrow Faster(x, y)))$

3. $\forall x (Greyhound(x) \rightarrow Dog(x))$

4. $\forall x \forall y \forall z (Faster(x, y) \wedge Faster(y, z) \rightarrow Faster(x, z))$

5. $Rabbit(Bunny)$

6. $Horse(Harry)$

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7. $Greyhound(Arrow) \wedge \forall y (Rabbit(y) \rightarrow Faster(Arrow, y))$

- 2, EInst

Proof example (cont.)

$8. Greyhound(Arrow)$ - 7, CElim

$9. \forall y(Rabbit(y) \rightarrow Faster(Arrow, y))$ - 7, CElim

$10. Rabbit(Bunny) \rightarrow Faster(Arrow, Bunny))$ - 9, UInst

$11. Faster(Arrow, Bunny)$ - 10, 5, MP

$12. Greyhound(Arrow) \rightarrow Dog(Arrow)$ - 3, UInst

$13. Dog(Arrow)$ - 12, 8, MP

$14. Horse(Harry) \wedge Dog(Arrow) \rightarrow Faster(Harry, Arrow)$

- 1, UInst

Proof example (cont.)

$15.\ Horse(Harry) \wedge Dog(Arrow)$ - 6, 13, CIntr

$16.\ Faster(Harry, Arrow)$ - 14, 15, MP

$17.\ Faster(Harry, Arrow) \wedge Faster(Arrow, Bunny) \rightarrow Faster(Harry, Bunny)$ - 4, UInst

$18.\ Faster(Harry, Arrow) \wedge Faster(Arrow, Bunny)$ - 16, 11, CIntr

$19.\ Faster(Harry, Bunny)$ - 17, 18, MP

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Normal forms

- Conjuctive normal form (CNF)

$$\Phi = \phi_1 \wedge \ldots \wedge \phi_n$$

$$\phi_i = \phi_{i1} \vee \ldots \vee \phi_{in_i}$$

- Disjunctive normal form (DNF)

$$\Phi = \phi_1 \vee \ldots \vee \phi_n$$

$$\phi_i = \phi_{i1} \wedge \ldots \wedge \phi_{in_i}$$

- Any formula can be converted to an equivalent formula in normal form.

Normal forms for PL

- Remove implications and equivalences:

$$(p \leftrightarrow q) \leftrightarrow ((p \to q) \wedge (q \to p))$$
$$(p \to q) \leftrightarrow (\neg p \vee q)$$

- Move negation toward propositions

$$\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$$
$$\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$$

- Reach desired normal form via distributivity of conjunction and disjunction

Clausal form

- Disjunction of literals
- Several notations:
  - Disjunction: $\neg P(x) \vee Q(x)$
  - Set: $\{\neg P(x), Q(x)\}$
  - Consequence: $P(x) \Rightarrow Q(x)$
  - Prolog rule: $Q(x) :- P(x)$
- Horn clause = at most one positive literal
- Definite clause = one positive literal
- Fact = one positve, no negative

Clausal form for FOPL

- Remove implications and equivalences
- Move quantifiers at the beginning

$$Qx\, F[x] \vee G \leftrightarrow Qx\, (F[x] \vee G)$$

$$Qx\, F[x] \wedge G \leftrightarrow Qx\, (F[x] \wedge G)$$

$$\neg \forall x\, F[x] \leftrightarrow \exists x\, (\neg F[x])$$

$$\neg \exists x\, F[x] \leftrightarrow \forall x\, (\neg F[x])$$

$$\forall x\, F[x] \wedge \forall x\, H[x] \leftrightarrow \forall x (F[x] \wedge H[x])$$

$$\exists x\, F[x] \vee \exists x\, H[x] \leftrightarrow \exists x\, (F[x] \vee H[x])$$

$$Q_1 x\, F[x] \wedge Q_2 x\, H[x] \leftrightarrow Q_1 x\, Q_2 z\, (F[x] \wedge H[z])$$

$$Q_1 x\, F[x] \vee Q_2 x\, H[x] \leftrightarrow Q_1 x\, Q_2 z\, (F[x] \vee H[z])$$

- Move negations toward predicates

Clausal form for FOPL

- Remove quantifiers
  - Variables bound to existential quantifiers are replaced by functions that take variables of all preceding universal quantifiers as arguments, then the corresponding existential quantifiers are dropped
  - Universal quantifiers are then removed
- Use distributivity and associativity to get CNF
- Split conjunctions to get clauses

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