Propositional Logic, SAT, NP-complete problems

Algorithms and Complexity Theory

Matei Popovici[1]

[1]POLITEHNICA University of Bucharest
Computer Science and Engineering Department, Bucharest, Romania

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Abbreviations:

- $\varphi \rightarrow \psi$ is $\neg\varphi \vee \psi$
- $\varphi \leftrightarrow \psi$ is $\varphi \rightarrow \psi \wedge \psi \rightarrow \varphi$

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We also write $I = \{x \leftarrow 0\}$ instead of $I(x) = 0$.

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Definition (Validity)

We say a formula φ is **valid** iff for all I, $I \models \varphi$

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Proposition

Every formula φ is equivalent to a formula in CNF form.

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Let φ be a formula in CNF form. The problem of deciding whether φ is **satisfiable** is called SAT.

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Proof.

- Blackboard - □

Bibliography I

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