Computing Amortized Costs for Tables with Expansion and Contraction

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1 Tables with Expansion (only)

Let T be a dynamic table and S be a sequence of size n of operations of type `insert`. Let T_i designate the state of T after operation i, $elems(T_i)$ and $size(T_i)$ designate the number of elements in T and the size of T after operation i, respectively. We designate by T_0 the state of T before any operation in S is performed.

Once a table is full ($elems(T_i) = size(T_i)$), a subsequent insertion triggers the doubling of the table size.

We establish the **amortized cost** of an operation of type `insert` using the **potential method**. Let:

$$\Phi(T_i) = 2 \cdot elems(T_i) - size(T_i) \text{ for } i \in S$$
$$\Phi(T_0) = 0$$

It is easy to see that $\Phi(T_n) \geq 0$ for any sequence of n operations.

Case 1 ($elems(T_i)/size(T_i) < 1$) i.e. operation i does not trigger an expansion. Therefore, $size(T_i) = size(T_{i-1})$ and $elems(T_i) = 1 + elems(T_{i-1})$.

$$\begin{aligned}
\hat{c}_i &= c_i + \Phi(T_i) - \Phi(T_{i-1}) \\
&= 1 + 2 \cdot elems(T_i) - size(T_i) - 2 \cdot elems(T_{i-1}) + size(T_{i-1}) \\
&= 1 + 2 + 2 \cdot elems(T_{i-1}) - size(T_{i-1}) - 2 \cdot elems(T_{i-1}) + size(T_{i-1}) \\
&= 3
\end{aligned}$$

unde i desemneaza o operatie de tip `insert`.

Case 2 $(elems(T_{i-1})/size(T_{i-1}) = 1)$ i.e. operation i does trigger an expansion. Therefore, $c_i = elems(T_{i-1}) + 1$, $size(T_i) = 2 \cdot size(T_{i-1})$ and $elems(T_i) = 1 + elems(T_{i-1})$.

$$\begin{aligned}
\hat{c}_i &= c_i + \Phi(T_i) - \Phi(T_{i-1}) \\
&= 1 + 2 \cdot elems(T_i) - size(T_i) - 2 \cdot elems(T_{i-1}) + size(T_{i-1}) \\
&= elems(T_{i-1}) + 1 + 2 + 2 \cdot elems(T_{i-1}) - 2 \cdot size(T_{i-1}) - 2 \cdot elems(T_{i-1}) + size(T_{i-1}) \\
&= elems(T_{i-1}) + 3 - size(T_{i-1}) \\
&= 3
\end{aligned}$$

2 Tables with Expansion and Contraction

Consider now a sequence S of n operations of types `insert` and `remove`, and the following potential function:

$$\Phi(T_i) = \begin{cases} 2 \cdot elems(T_i) - size(T_i) & \text{if } elems(T_i)/size(T_i) \geq 1/2 \\ size(T_i)/2 - elems(T_i) & \text{if } elems(T_i)/size(T_i) < 1/2 \\ 0 & \text{if } i = 0 \end{cases}$$

If a removal i makes $elems(T_i) = size(T_i)/4$, then the table T is contracted: $size(T_i) = size(T_{i-1})/2$.

The amortized cost $\hat{c}_i$ where i is an operation of type `insert` follows the same cases as above, when $elems(T_i)/size(T_i) \geq 1/2$.

Case 3.a $(elems(T_i)/size(T_i) = 1/2)$. Furthermore $elems(T_i) = elems(T_{i-1}) + 1$ and $size(T_i) = size(T_{i-1})$

$$\begin{aligned}
\hat{c}_i &= c_i + \Phi(T_i) - \Phi(T_{i-1}) \\
&= 1 + 2 \cdot elems(T_i) - size(T_i) - size(T_{i-1})/2 + elems(T_{i-1}) \\
&= 1 + size(T_i) - size(T_i) - size(T_i)/2 + elems(T_i) - 1 \\
&= 0 - size(T_i)/2 + elems(T_i) \\
&= 0
\end{aligned}$$

Case 4.a $(elems(T_i)/size(T_i) < 1/2)$ The same setting as in Case 3.a holds.

$$\begin{aligned}
\hat{c}_i &= c_i + \Phi(T_i) - \Phi(T_{i-1}) \\
&= 1 + size(T_i)/2 - elems(T_i) - size(T_{i-1})/2 + elems(T_{i-1}) \\
&= 1 - elems(T_i) + elems(T_i) - 1 \\
&= 0
\end{aligned}$$

Notice that, although in cases 3.a and 4.a the amortized cost for `insert` is not equal to 3, it is however bounded-above by 3, therefore 3 is an appropriate value for $\hat{c}_i = 3$, for i of type `insert`.

The amortized cost $\hat{c}_i$ where i is an operation of type **`remove`** follows the cases:

Case 1.b $(elems(T_{i-1})/size(T_{i-1}) < 1/2)$. Moreover, we do not have a contraction. We have $elems(T_i) = elems(T_{i-1}) - 1$ and $size(T_i) = size(T_{i-1})$.

$$\begin{aligned}
\hat{c}_i &= c_i + \Phi(T_i) - \Phi(T_{i-1}) \\
&= 1 + size(T_i)/2 - elems(T_i) - size(T_{i-1})/2 + elems(T_{i-1}) \\
&= 1 - elems(T_i) + elems(T_{i-1}) \\
&= 2
\end{aligned}$$

Case 2.b $(elems(T_{i-1})/size(T_{i-1}) < 1/2)$. Also, we have a contraction. Thus $elems(T_i) = elems(T_{i-1}) - 1$, $size(T_i) = size(T_{i-1})/2$ and $elems(T_i) = size(T_i)/2$.

$$\begin{aligned}
\hat{c}_i &= c_i + \Phi(T_i) - \Phi(T_{i-1}) \\
&= 1 + elems(T_i) + size(T_i)/2 - elems(T_i) - size(T_{i-1})/2 + elems(T_{i-1}) \\
&= 1 + elems(T_i) + elems(T_i) - elems(T_i) - 2 \cdot elems(T_i) + elems(T_i) + 1 \\
&= 2
\end{aligned}$$

Case 2.c $(elems(T_{i-1})/size(T_{i-1}) = 1/2)$. The same conditions as in case 1.b hold.

$$\begin{aligned}
\hat{c}_i &= c_i + \Phi(T_i) - \Phi(T_{i-1}) \\
&= 1 + size(T_i)/2 - elems(T_i) - 2 \cdot elems(T_{i-1}) + size(T_{i-1}) \\
&= 1 + elems(T_{i-1}) - elems(T_{i-1}) + 1 - 2 \cdot elems(T_{i-1}) + 2 \cdot elems(T_{i-1}) \\
&= 2
\end{aligned}$$

Case 2.d $(elems(T_i)/size(T_i) > 1/2)$. Again, the same conditions as in case 1.b hold.

$$\begin{aligned}
\hat{c}_i &= c_i + \Phi(T_i) - \Phi(T_{i-1}) \\
&= 1 + 2 \cdot elems(T_i) - size(T_i) - 2 \cdot elems(T_{i-1}) + size(T_{i-1}) \\
&= 1 + 2 \cdot elems(T_{i-1}) - 2 - 2 \cdot elems(T_{i-1}) \\
&= -1
\end{aligned}$$

Notice that the amortized cost for `remove` is bounded above by 2, therefore, we can set $\hat{c}_i = 2$ when i is of type `remove`.