Complexity Classes and Polynomial Reductions

Algorithms and Complexity Theory

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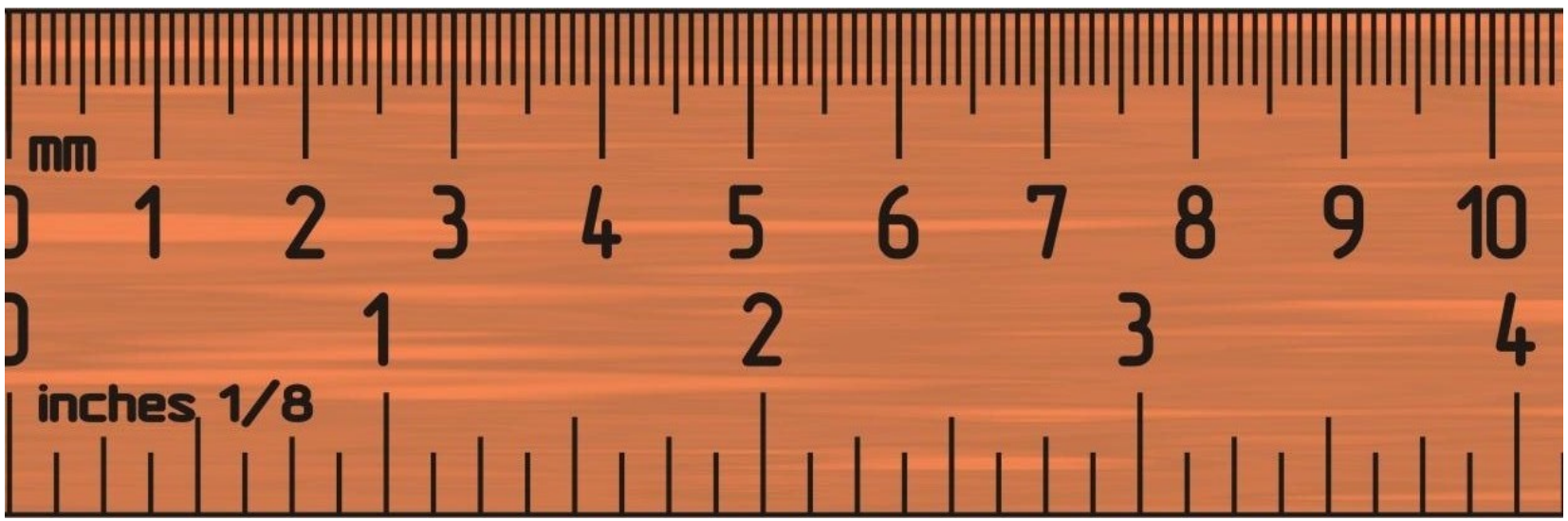
*Is it possible to construct a **hierarchy** of problems, based on hardness ?*

A hierarchy of complexity classes

P



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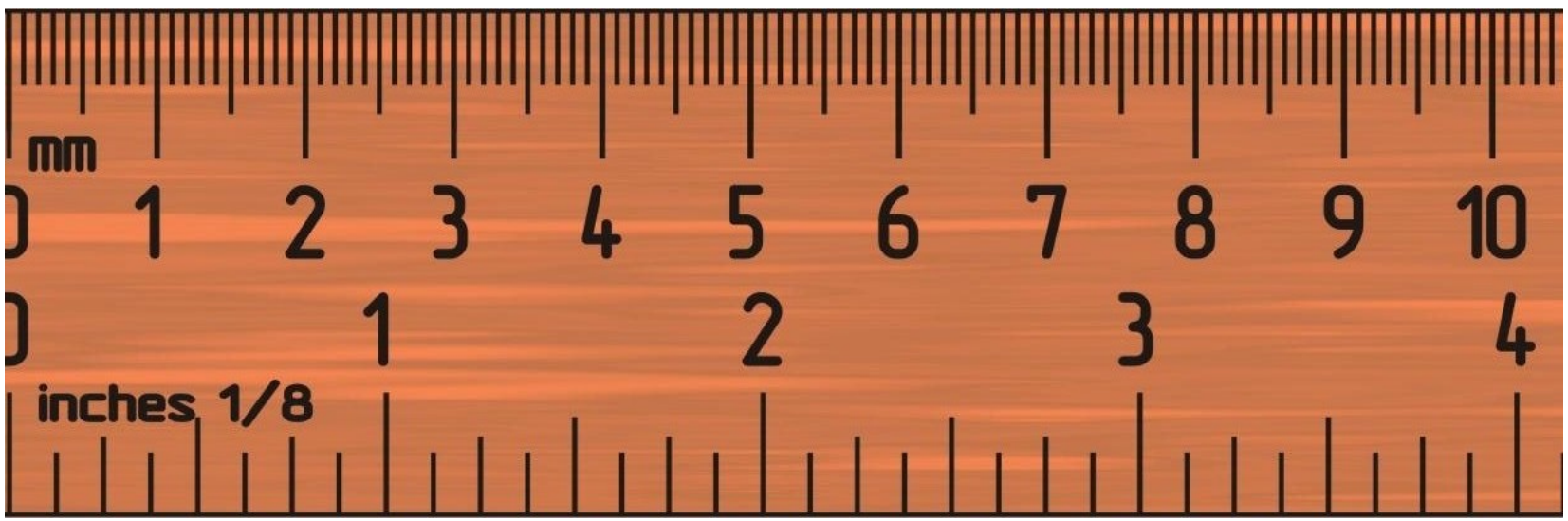

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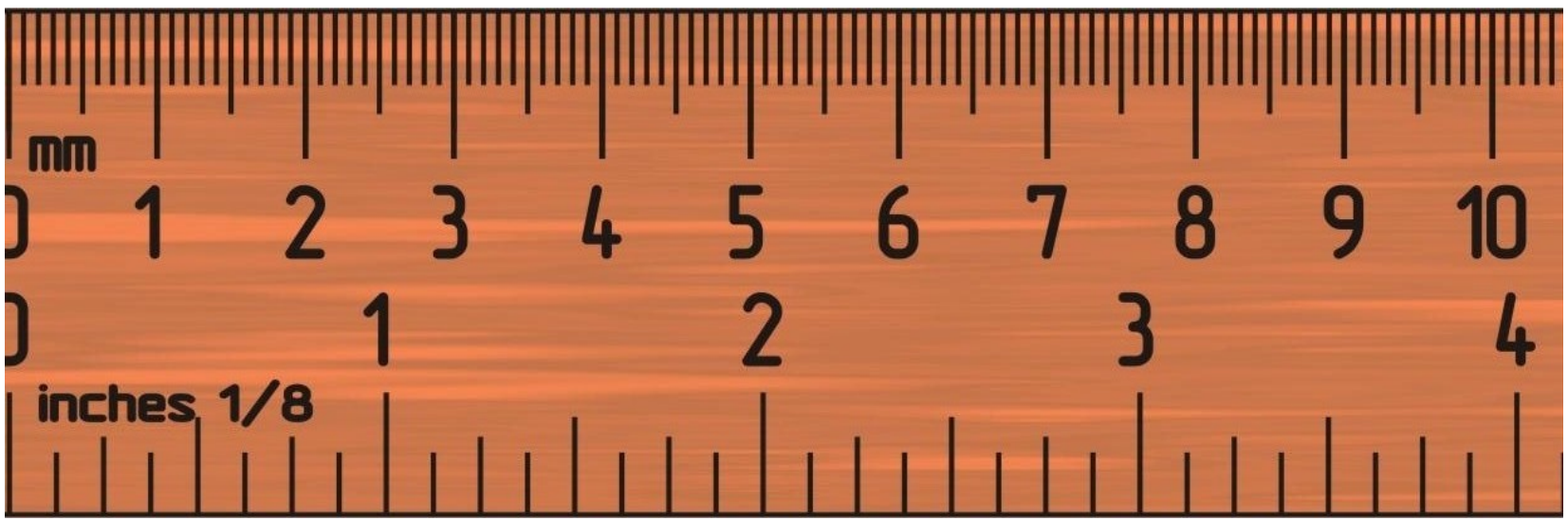

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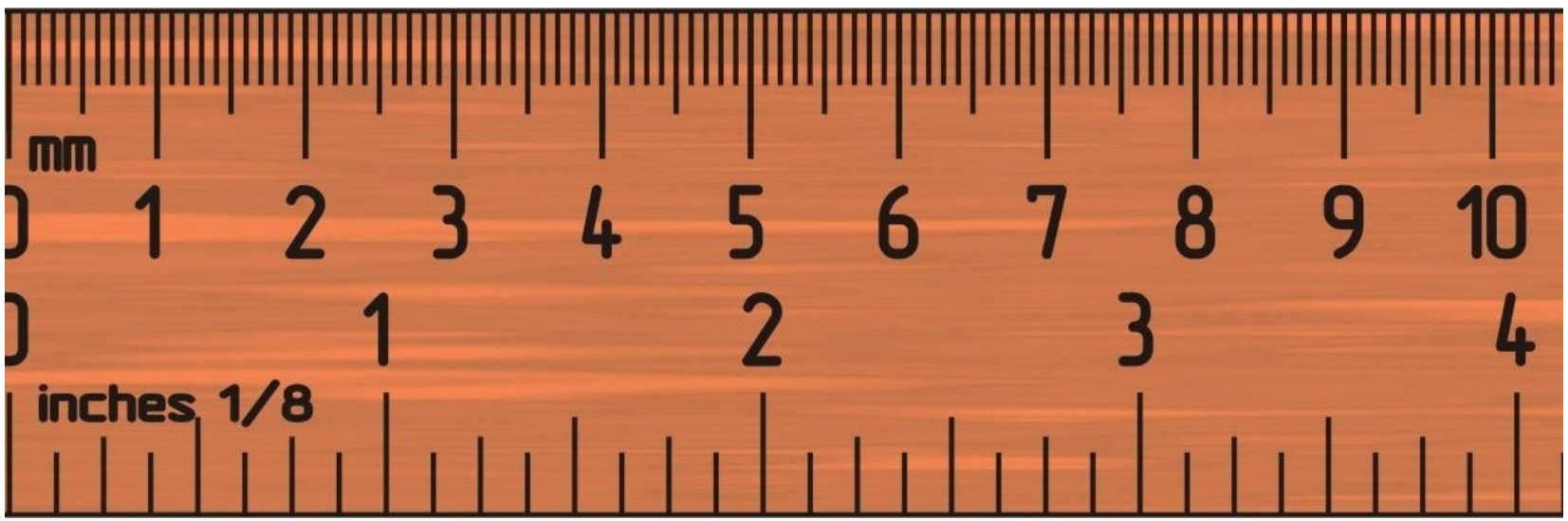

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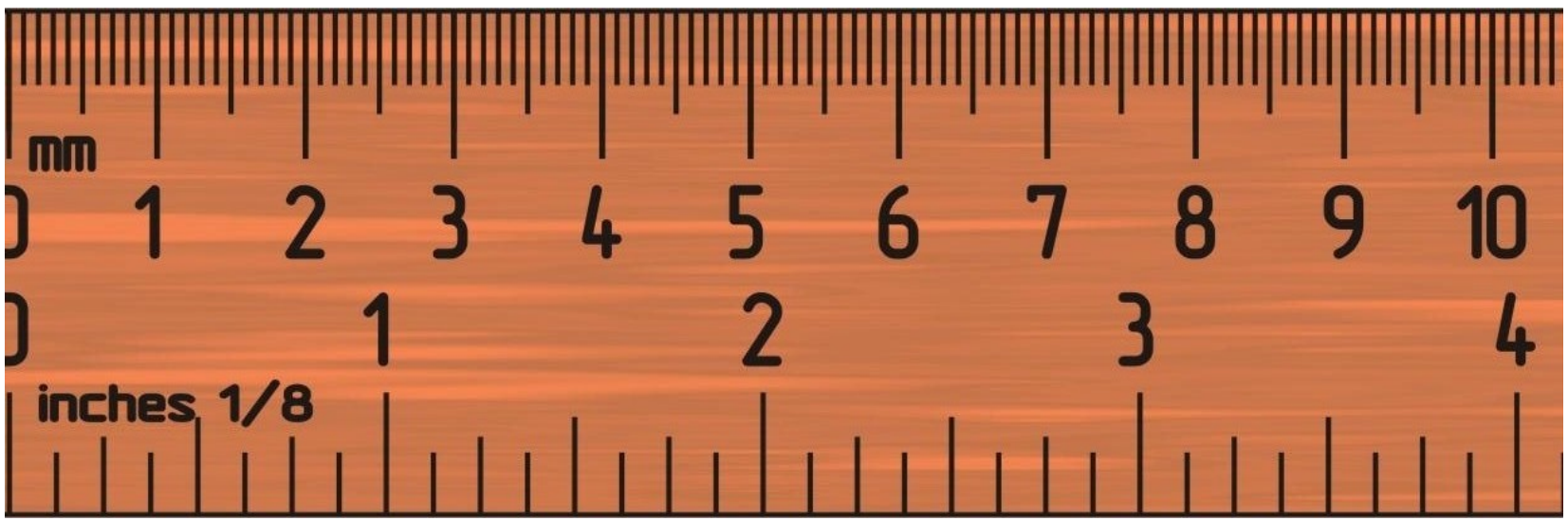

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- to answer *yes* to this problem means to answer *no* to its complement

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    - Planning problems in: HPC systems, air space control, etc. are feasible.

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- Given a problem **Q** and a complexity class X such that $\mathbf{Q} \in X$, then X is an upper bound on the difficulty of **Q**
- How can we provide with lower bounds for **Q** ?

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$Q_2 \in P$ and $Q_1 \leq_p Q_2$, then $Q_1 \in P$.

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Proposition (Tool for proving hardness)

Let Q, Q' be two problems such that Q' is X-hard and $Q' \leq_p Q$. Then Q is X-hard.

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- Let Q be an NP-hard problem.

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- Finding lower bounds for a problem $Q \equiv$ proving **hardness** w.r.t. a complexity class X.

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I conjecture that there is no good algorithm for the travelling salesman problem. My reasons are the same as for any mathematical conjecture: (i) it is a legitimate mathematical possibility, and (ii) I do not know. (Jack Edmonds, 1966)

*There exist problems which **cannot be solved efficiently** ?*

- How do we formally express that a problem is **solvable** ?
- Is it the case that, *finding the correct answer* is harder than *checking if a **given answer is correct*** ?
- How do we express that a problem is **harder** than another problem ?
- Is it possible to construct a **hierarchy** of problems, based on hardness ?

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