Asymptotic Notations outside Computer Science

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1 An example

Consider the following function:

$$f(x) = \frac{1}{1-x}$$

For values of x which are strictly smaller than 1, we have that:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{i=0}^{\infty} = x^i$$

We notice that, in the sequence $\{x^i\}_{i \in \mathbb{N}}$, for each i, $x^{i+1} \in o(x^i)$, since $x < 1$. Thus, one way to write $f(x)$ is:

$$\frac{1}{1-x} = 1 + x + o(x) + o(x^2) + o(x^3) + \ldots = 1 + O(x)$$

The above formulation shows that we can approximate $f(x)$ by $1 + c \cdot x$, where c is some strictly positive constant (which is unknown). The approximation is rather poor. A better approximation may yeild from:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + O(x^4)$$

Here, we approximate $f(x)$ by $1 + x + x^2 + x^3$ plus some function $c \cdot x^4$. We may consider the unknown function $c \cdot x^4$ as the *approximation error*. Thus, by using $1 + x + x^2 + x^3$ instead of $f(x)$, we may be *off* by $O(x^4)$, where $x < 1$.

2 Asymptotic expansions

More generally, given a sequence $\{g_i(x)\}_{i \in \mathbb{N}}$, such that, for each $i \in \mathbb{N}$, $g_{i+1}(x) = o(g_i(x))$, and a function f which can be written as a series:

$$f(x) = c_0 \cdot g_0(x) + c_1 \cdot g_1(x) + c_2 \cdot g_2(x) + \dots$$

then, f can be written as any of the following **asymptotic expansions** (also known as Pointcaré expansions):

$$\begin{array}{l} f(x) = O(g_0(x)) \\ f(x) = c_0 \cdot g_0(x) + O(g_1(x)) \\ f(x) = c_0 \cdot g_0(x) + c_0 \cdot g_1(x) + O(g_2(x)) \\ \dots \end{array}$$

Notice that each expansion $i + 1$ is preferred over i, since it exposes the constant c_i, thus providing a finer approximation of f.