# Amortised Analysis

## Algorithms and Complexity Theory

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October 26, 2012

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- $c_i^S$ (or just $c_i$ when $S$ is understood from the context): *the cost of the i-th operation in S*

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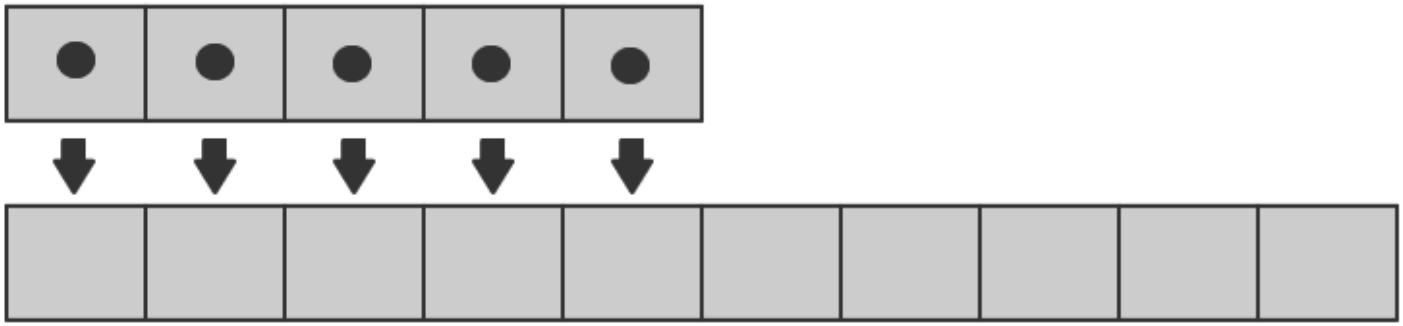

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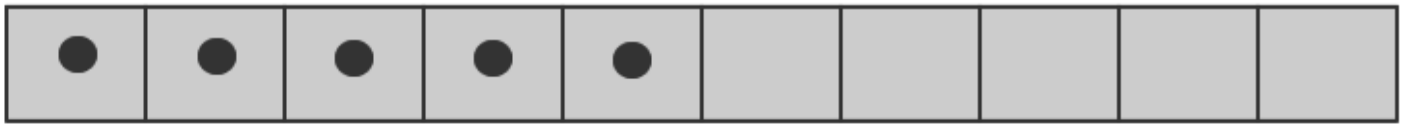

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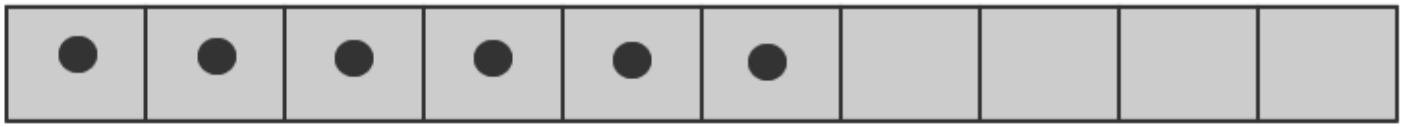

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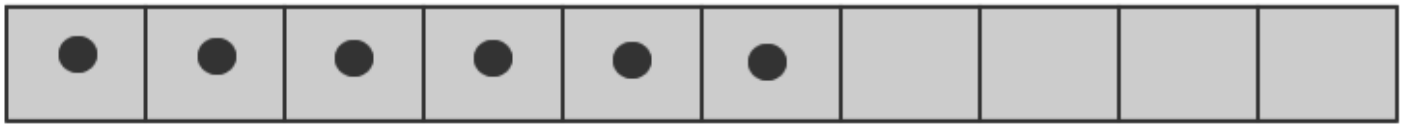

The cost of operation $i$ is: $c_i = k + 1$ where $k$ is the number of elements in $T$

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# More assumptions

Let $T$ be a table and $S$ be a sequence of operations:

- we consider $T$ to be of **capacity/size 0**, before running $S$.
- deleting an element from $T$ when it is empty has cost 1 and leaves $T$ unchanged.

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- The approximation $O(n^2)$ is **poor** !

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Let us assume that $S$ is a sequence with **only insertion operations**
Then:

$$c_i = \begin{cases} i & \text{if } i-1 \text{ is a power of 2} \\ 1 & \text{otherwise} \end{cases}$$

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- we do not distinguish between **types** of operations

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# Bibliography I


Thomas H. Cormen, Clifford Stein, Ronald L. Rivest, and Charles E. Leiserson.
*Introduction to Algorithms*.
McGraw-Hill Higher Education, 2nd edition, 2001.